

# Action principles for higher and fractional spin gravities

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## Abstract

We review various off-shell formulations for interacting higher-spin systems in dimensions 3 and 4. Associated with higher-spin systems in spacetime dimension 4 is a Chern–Simons action for a superconnection taking its values in a direct product of an infinite-dimensional algebra of oscillators and a Frobenius algebra. A crucial ingredient of the model is that it elevates the rigid closed and central two-form of Vasiliev’s theory to a dynamical 2-form and doubles the higher-spin algebra, thereby considerably reducing the number of possible higher spin invariants and giving a nonzero effective functional on-shell. The two action principles we give for higher-spin systems in 3D are based on Chern–Simons and BF models. In the first case, the theory we give unifies higher-spin gauge fields with fractional-spin fields and an internal sector. In particular, Newton’s constant is related to the coupling constant of the internal sector. In the second case, the BF action we review gives the fully nonlinear Prokushkin–Vasiliev, bosonic equations for matter-coupled higher spins in 3D. We present the truncation to a single, real matter field relevant in the Gaberdiel–Gopakumar holographic duality. The link between the various actions we present is the fact that they all borrow ingredients from Topological Field Theory. It has been conjectured that there is an underlying and unifying 2-dimensional first-quantised description of the previous higher-spin models in 3D and 4D, in the form of a Cattaneo–Felder-like topological action containing fermionic fields.

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# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Frobenius-Chern-Simons Action for 4D Higher Spin Gravity</b>	<b>2</b>
2.1	Base manifold . . . . .	3
2.2	Superconnection and Frobenius-Chern-Simons action . . . . .	5
2.3	Component formulation . . . . .	8
2.4	Comparison with the duality extended Vasiliev system . . . . .	9
2.5	On-shell actions from topological invariants . . . . .	10
2.6	Linearized Fluctuations . . . . .	11
<b>3</b>	<b>Fractional spin gravity theory</b>	<b>13</b>
3.1	The fractional spins algebra . . . . .	14
3.2	Gravitational and gauge couplings from Chern-Simons fractional spin gravity . . . . .	17
<b>4</b>	<b>Matter-coupled 3D higher-spin gravity</b>	<b>20</b>
4.1	Geometric Formulation of Prokushkin–Vasiliev’s system . . . . .	20
4.2	Covariant Hamiltonian action . . . . .	26
4.3	Consistent truncations . . . . .	29
<b>5</b>	<b>Higher spins and topological strings</b>	<b>31</b>

## 1 Introduction

For our contribution to the proceedings of the International Workshop on Higher Spin Gauge Theories that took place in Singapore on 5–7 November 2015, we review a number of results presented in Refs. [1, 2, 3, 4] where off-shell formulations of nonlinear higher-spin systems have been found that describe interacting higher-spin fields in spacetime dimensions 3 and 4.

In Section 2 we first review the work [3] giving a Frobenius–Chern–Simons model for nonlinear higher-spin theory in 4D. This model makes use of ingredients provided in the geometrical formulation of higher spin gravity [5] and the action principle proposed in Ref. [6]. The important new properties are the introduction of a dynamical 2-form and the attendant phenomenon of the higher spin algebra doubling. This leads to a more predictive power, as it restricts the possible higher spin invariant functionals.

In Section 3 based on Ref. [2], we review the construction of 3D higher-spin (HS) models coupled to an internal  $U(\infty) \otimes U(\infty)$  sector and fractional-spin fields. The latter fields generalise the gravitini and our model can be seen as an extension of the Achucaro–Townsend Chern–Simons supergravity [7]. In particular, the coupling constant in the  $U(\infty) \otimes U(\infty)$  internal sector is proportional to the gravitational Newton constant.

In Section 4 based on Ref. [4], we review the construction of an action that reproduces the fully nonlinear and bosonic Prokushkin–Vasiliev equations [8]. The action is shown to restrict to the Blencowe action [9, 10]

thereby reproducing the standard kinetic terms in the higher-spin sector of the action. We also review from Ref. [4] the unfolded nonlinear equations for 3D HS fields coupled to a single real scalar field in the matter sector, as this is relevant in the context of the Gaberdiel–Gopakumar duality [11, 12].

Finally, in Section 5 we review the works [1, 13] where a topological open-string model of the Cattaneo–Felder type [14] was proposed as an underlying first-quantised model for higher-spin gauge theory.

## 2 Frobenius-Chern-Simons Action for 4D Higher Spin Gravity

An outstanding problem in higher spin gravity in four (and higher) dimensions [15, 16, 17] is to find an action principle with desirable properties. Treating this problem as nonlinear completion of Fronsdal kinetic terms in a Noether procedure approach runs into considerable technical difficulties. Indeed, in the metric-like [18] and the related frame-like [19, 20, 21] approaches, long term efforts — see for example [22, 23] and [24, 25, 26, 27, 28, 29, 30, 31, 32] in the case of  $AdS$  background — has so far led to primarily cubic interactions. Beyond the cubic order, the fact that higher spin gravity has a mass scale set by the bare cosmological constant while nonabelian higher spin symmetries require higher derivative vertices, lead to intractable abelian vertices built from curvatures and their higher derivatives; see [28] and the review [33].

The Noether procedure approach does not exploit the fact that Vasiliev’s equations [15, 16, 17] provide a fully non-linear description of higher spin gravity on-shell. Furthermore, it is background dependent procedure since it is based on perturbation around  $AdS_4$ . Both drawbacks can be avoided by considering covariant Hamiltonian actions from which the background independent full Vasiliev equations follow. These equations are Cartan integrable systems of differential forms on special noncommutative manifolds taking their values in associative higher spin algebras. Treating these forms as the fundamental fields following the AKSZ approach [34], one is led to a path integral formulation [35] based on covariant Hamiltonian actions [6] on noncommutative manifolds with boundaries. The importance of the boundaries, which are absent in the related proposals [36, 37], is that they facilitate the deformation of the bulk action by boundary terms [5] that contribute to the action but not its variation on-shell. The resulting higher spin amplitudes reproduce desired holographic correlation functions [38, 39, 40], which are suggestive of an underlying topological open string [41, 1, 42]. However, the presence of a large number of free parameters impede the predictive power of the model.

A more predictive model has been proposed in Ref. [3] in which the closed and central holomorphic two form in Vasiliev equations is elevated to a dynamical two-form master field, and the higher spin algebra is necessarily extended to include new master one-form field. The model also employs an eight dimensional Frobenius algebra and provides an action which takes the form of a Chern–Simons term for a superconnection that accommodates all the master fields. For this reason the model is referred to as the Frobenius–Chern–Simons (FCS) gauge theory. The construction of the model has been described in detail in Ref. [3]. Here we shall summarize its salient features.

## 2.1 Base manifold

The model is formulated in terms of differential forms on the direct product space  $\mathcal{M}_9 = \mathcal{X}_5 \times \mathcal{Z}_4$ , where  $\mathcal{X}_5$  is a five-dimensional commutative manifold with boundary  $\mathcal{X}_4 = \partial\mathcal{X}_5$ , containing the original spacetime manifold  $\mathcal{M}_4$  as a possibly open subset, and  $\mathcal{Z}_4$  is a four-dimensional noncommutative space without boundary. Thus,  $\partial\mathcal{M}_9 = \mathcal{X}_4 \times \mathcal{Z}_4$ , where  $\mathcal{X}_5 = \mathcal{X}_4 \times [0, \infty[$ .

The topology of  $\mathcal{Z}_4$  may be chosen in a variety ways with nontrivial and interesting consequences to be investigated. In Ref. [3],  $\mathcal{Z}_4$  is obtained from the standard noncommutative  $\mathbb{C}^4$  by choosing a real form and a compatible convolution formula for the star product and then adding points at infinity to create a compact noncommutative space that can be used to define a (graded cyclic) trace operation. Moreover,  $\mathcal{Z}_4$  is taken to be closed to avoid boundary terms, and that its structure admits a certain closed two-form and a global  $SL(2; \mathbb{C})$  symmetry, in order to make contact with Vasiliev's theory. To this end, one introduces canonical coordinates  $(z^\alpha, \bar{z}^{\dot{\alpha}})$  ( $\alpha, \dot{\alpha} = 1, 2$ ) and anti-commuting differentials  $(dz^\alpha, d\bar{z}^{\dot{\alpha}})$  on  $\mathbb{C}^4$ . We then consider a formally defined associative star product algebra given by the space  $\Omega(\mathbb{C}^4)$  of differential forms equipped with two associative composition rules, namely the standard graded commutative wedge product rule, denoted by juxtaposition, and the graded noncommutative rule

$$f \star g = f \exp \left( -i(\overleftarrow{\partial}^\alpha \overrightarrow{\partial}_\alpha + \overleftarrow{\partial}^{\dot{\alpha}} \overrightarrow{\partial}_{\dot{\alpha}}) \right) g, \quad (2.1)$$

The star product is thus the representation using Weyl ordering symbols of the associative algebra of composite operators built from anti-commuting line elements and noncommutative coordinates with canonical commutation rules

$$[z^\alpha, z^\beta]_\star = -2i\epsilon^{\alpha\beta}, \quad [z^\alpha, \bar{z}^{\dot{\alpha}}]_\star = 0, \quad [\bar{z}^{\dot{\alpha}}, \bar{z}^{\dot{\beta}}]_\star = -2i\epsilon^{\dot{\alpha}\dot{\beta}}. \quad (2.2)$$

In models with four-dimensional Lorentz symmetry, it is natural to select real forms on the real slice

$$\mathcal{R}_4^{\mathbb{C}} = \{(z^\alpha, \bar{z}^{\dot{\alpha}}) : (z^\alpha)^\dagger = -\bar{z}^{\dot{\alpha}}, \quad (\bar{z}^{\dot{\alpha}})^\dagger = -z^\alpha\} \cong \mathbb{C}^2 \times \overline{\mathbb{C}^2} \subset \mathbb{C}^4, \quad (2.3)$$

on which  $z^\alpha$  is thus a complex doublet.

In order to include Gaussian elements and distributions, it is useful to first introduce auxiliary integral representations of the star product (2.1) as follows

$$f \star g = \int_{\mathcal{R}_4^{\mathbb{R}}} \frac{d^2\xi d^2\tilde{\xi}}{(2\pi)^2} \int_{\mathcal{R}_4^{\mathbb{R}}} \frac{d^2\eta d^2\tilde{\eta}}{(2\pi)^2} e^{i(\eta^\alpha \xi_\alpha + \tilde{\eta}^{\dot{\alpha}} \tilde{\xi}_{\dot{\alpha}})} f(z + \xi, \bar{z} + \tilde{\xi}; dz, d\bar{z}) g(z - \eta, \bar{z} - \tilde{\eta}; dz, d\bar{z}), \quad (2.4)$$

where the integration domain is chosen conveniently as the real

$$\mathcal{R}_4^{\mathbb{R}} = \{(\xi^\alpha, \tilde{\xi}^{\dot{\alpha}}) : \xi^\alpha, \tilde{\xi}^{\dot{\alpha}} \in \mathbb{R}^2\} \cong \mathbb{R}^2 \times \mathbb{R}^2 \quad (2.5)$$

The graded cyclic trace operation on  $\Omega(\mathcal{Z}_4)$  is defined as

$$\text{STr}_{\Omega(\mathcal{Z}_4)} f = \int_{\mathcal{R}_4^{\mathbb{R}}} f, \quad (2.6)$$

which projects onto the top form in  $f$ . Thus, if  $f$  is a top form in  $\Omega(\mathcal{Z}_4)$  then its representative in  $\Omega(\mathcal{R}_4^{\mathbb{R}})$  must fall off sufficiently fast at infinity for the integral to be convergent. Therefore,  $\mathcal{Z}_4$  must be a compact manifold obtained by adding points to  $\mathcal{R}_4^{\mathbb{R}}$  at infinity to extend its differential Poisson algebra structure [43, 44, 45, 46] (see also [1, 13]). This can be achieved by assuming that  $\mathcal{Z}_4$  admits a Poisson structure and a compatible pre-connection. The latter is assumed to be trivial for simplicity. In addition,  $\mathcal{Z}_4$  is required to be closed, such that  $\text{STr}_{\Omega(\mathcal{Z}_4)} df = 0$ , in order to avoid boundary terms from  $\mathcal{Z}_4$  in varying the FCS action. Assuming that  $f$  and  $g$  are two smooth symbols that fall off sufficiently fast, it follows from (2.4) that

$$\text{STr}_{\Omega(\mathcal{Z}_4)} f \star g = \int_{\mathcal{R}_4^{\mathbb{R}}} f \star g = \int_{\mathcal{R}_4^{\mathbb{R}}} fg , \quad (2.7)$$

which is graded cyclic. As this property will be useful in analyzing boundary conditions in the FCS model arising from the variational principle, we shall assume that (2.7) holds for all elements in  $\Omega(\mathcal{Z}_4)$  including distributions. Finally, in order to obtain Vasiliev's equations from the FCS action, it is assumed that  $\Omega(\mathcal{Z}_4)$  admits global  $SL(2; \mathbb{C})$  symmetry and contains the (globally defined) closed two-forms

$$j_z = -\frac{i}{4} dz^\alpha dz_\alpha \kappa_z , \quad \bar{j}_{\bar{z}} = (j_z)^\dagger , \quad (2.8)$$

where the inner Klein operator

$$\kappa_z = 2\pi \delta^2(z^\alpha) . \quad (2.9)$$

A choice of topology that satisfies all of the requirements as stated above is given by<sup>1</sup>

$$\Omega(\mathcal{Z}_4) = \bigoplus_{m, \bar{m}=0,1} (\Omega(S^2) \star (j_z)^{\star m}) \otimes (\Omega(S^2) \star (\bar{j}_{\bar{z}})^{\star \bar{m}}) , \quad (2.10)$$

where  $\Omega(S^2)$  consists of globally defined forms on  $S^2$  with Poisson structure obtained by extending the Poisson structure of (2.1) to the point at  $\infty$ . At this point, the resulting Poisson bivector and all its derivatives vanish. Hence, provided it is possible to exchange the order of differentiation and summation in (2.1) and using the fact that increasing number of derivatives of a form that falls off yields forms that fall off even faster, it follows that if  $f, g \in \Omega(S^2)$  then  $(f \star g)|_\infty = f|_\infty g|_\infty$  i.e. the point at infinity is a commuting point of  $\Omega(S^2)$ . In other words, one is working with a topological two-sphere equipped with a differential Poisson algebra with trivial pre-connection. Moreover, in order for the elements in  $\Omega(\mathcal{Z}_4)$  to have finite traces, it is assumed that the top forms on each two-sphere fall off sufficiently fast at infinity working in the original  $\mathbb{R}^2 \times \mathbb{R}^2$  coordinate chart. For this fall-off condition to be embeddable in a differential star product algebra, also the forms in lower degrees must fall off appropriately at infinity. In particular, the only forms that can have finite values at infinity are the zero-forms. Thus, in effect, one has

$$\mathcal{Z}_4 = S^2 \times S^2 , \quad (2.11)$$

by assuming boundary conditions at the commuting points at infinity and allowing for delta function distributions at the origins so as to create a space of forms that is closed under exterior differentiation and star

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<sup>1</sup>The manifolds  $S^4$  and  $S^3 \times S^1$  do not admit  $j$ , while  $T^2 \times T^2$  breaks global  $SL(2; \mathbb{C})$  symmetry.

products, and has a space of top forms with finite traces that vanish for exact elements and obey (2.7). For this to hold true it is important that the delta function  $\kappa_z$  always appears together with line elements in the combination  $j_z$  given in (2.8), which obeys  $j_z \star j_z = 0$ , whereas the inclusion of  $\kappa_z$  into the algebra would require the inclusion of  $j_z \star \kappa_z$  as well which is not integrable.

## 2.2 Superconnection and Frobenius-Chern-Simons action

The construction of the FCS action employs an eight-dimensional, 3-graded Frobenius algebra

$$\mathcal{F} = \mathcal{F}^{(-1)} \oplus \mathcal{F}^{(0)} \oplus \mathcal{F}^{(+1)} , \quad (e_{ij}, h e_{ij}) \in \mathcal{F}^{(i-j)} , \quad (2.12)$$

where  $e_{ij}$  is the  $2 \times 2$  matrix whose only non-vanishing entry is a 1 at the  $i$ th row and  $j$ th column, and  $h$  is a Klein element satisfying  $[h, e_{11}] = 0 = [h, e_{22}]$ ,  $\{h, e_{12}\} = 0 = \{h, e_{12}\}$ . The coordinate-like master fields are assembled into

$$X = \sum_{i,j} X^{ij} e_{ij} = \begin{pmatrix} A & B \\ \tilde{B} & \tilde{A} \end{pmatrix} , \quad (2.13)$$

and the momentum-like master fields, which will play the role of Lagrange multipliers, into

$$P = \sum_{i,j} P^{ij} e_{ij} = \begin{pmatrix} V & U \\ \tilde{U} & \tilde{V} \end{pmatrix} . \quad (2.14)$$

The above master fields are duality extended in the sense that they are formal sums of forms with degrees

$$\begin{aligned} \deg(B, A, \tilde{A}, \tilde{B}) &= \{(2n, 1 + 2n, 1 + 2n, 2 + 2n)\}_{n=0,1,2,3} , \\ \deg(\tilde{U}, V, \tilde{V}, U) &= \{(8 - 2n, 7 - 2n, 7 - 2n, 6 - 2n)\}_{n=0,1,2,3} . \end{aligned} \quad (2.15)$$

One then proceeds by defining a superconnection and superdifferential [47] as

$$Z = hX + P , \quad (2.16)$$

respectively, which are thus objects with odd total degree given by form degree plus Frobenius 3-degree. The space  $\Omega(\mathcal{M}_9)$  of differential forms is equipped with two associative composition rules, namely the standard graded commutative wedge product rule, denoted by juxtaposition, and Weyl ordered star products of functions. The need for Weyl ordering is due to the property (2.7) that is crucial for the boundary conditions to make sense in a noncommutative set up. Once one has put any star-product expression in its factorized form in  $Y$  and  $Z$ , say  $F(Y) \star G(Z)$  equal to  $F(Y)G(Z)$  in Weyl order, one makes assumptions about the functional classes to which  $F(Y)$  and  $G(Z)$  belong. See Section 3.7 of [48] for details.

For reasons explained in detail in [3], the action function is expressed in terms of globally defined configurations in

$$\mathcal{E} = \Omega(\mathcal{X}_5) \otimes \frac{1}{2}(1 + \pi\bar{\pi}) [\Omega(\mathcal{Z}_4) \otimes \mathcal{A} \otimes \frac{1}{2}(1 + k \star \bar{k})] , \quad (2.17)$$

where

$$\mathcal{A} = \mathcal{F} \otimes \mathcal{W}_0 \otimes \mathcal{K} . \quad (2.18)$$

Furthermore,  $\mathcal{W}_0$  is an extended Weyl algebra

$$\mathcal{W}_0 = \bigoplus_{r, \bar{r}=0,1} \text{Aq}(2) \star (\kappa_y)^{\star r} \star (\bar{\kappa}_{\bar{y}})^{\star \bar{r}} \quad (2.19)$$

where  $\text{Aq}(2)$  consists of star polynomials in two complex doublets  $(y^\alpha, \bar{y}^{\dot{\alpha}})$ ,  $\alpha, \dot{\alpha} = 1, 2$ , obeying the oscillator algebra

$$[y^\alpha, y^\beta]_\star = 2i\epsilon^{\alpha\beta} \ , \quad [y^\alpha, \bar{y}^{\dot{\beta}}]_\star = 0 \ , \quad [\bar{y}^{\dot{\alpha}}, \bar{y}^{\dot{\beta}}]_\star = 2i\epsilon^{\dot{\alpha}\dot{\beta}} \ . \quad (2.20)$$

The inner Klein operators in Weyl order are defined as

$$\kappa_y := 2\pi\delta^2(y) \ , \quad \bar{\kappa}_{\bar{y}} := 2\pi\delta^2(\bar{y}) \ , \quad (2.21)$$

so that  $P \in \mathcal{W}_0$  obey

$$\kappa_y \star P \star \kappa_y = \pi_y(P) \ , \quad \bar{\kappa}_{\bar{y}} \star P \star \bar{\kappa}_{\bar{y}} = \bar{\pi}_{\bar{y}}(P) \ , \quad (2.22)$$

where  $\pi_y$  and  $\bar{\pi}_{\bar{y}}$  are inner automorphisms whose action in Weyl order is given by

$$\pi_y(y) = -y \ , \quad \bar{\pi}_{\bar{y}}(\bar{y}) = -\bar{y} \ , \quad (2.23)$$

leaving intact all the Klein operators. Thus, the generic elements of  $\mathcal{W}_0$  is of the form

$$P = \sum_{r, \bar{r}=0,1} P_{\alpha(n), \dot{\alpha}(\bar{n})}^{r, \bar{r}} (\kappa_y)^{\star r} \star (\bar{\kappa}_{\bar{y}})^{\star \bar{r}} y^{(\alpha_1 \star \dots \star y^{\alpha_n})} \star \bar{y}^{(\dot{\alpha}_1 \star \dots \star \bar{y}^{\dot{\alpha}_{\bar{n}}})} \ . \quad (2.24)$$

Turning to  $\mathcal{K}$ , it is the collection of outer Klein operators

$$\mathcal{K} = \{1, k, \bar{k}, k \star \bar{k}\} \ . \quad (2.25)$$

where for  $f \in \mathcal{E}$ , the adjoint action

$$k \star f \star k = \pi(f) \ , \quad \bar{k} \star f \star \bar{k} = \bar{\pi}(f) \ , \quad (2.26)$$

and the outer automorphisms  $\pi$  and  $\bar{\pi}$  with the only nontrivial actions

$$\pi(y, z) = (-y, -z) \ , \quad \bar{\pi}(\bar{y}, \bar{z}) = (-\bar{y}, -\bar{z}) \ . \quad (2.27)$$

Employing the ingredients summarized above, the following action has been proposed [3]

$$S = \int_{\mathcal{M}_9} \text{Tr}_{\mathcal{A}} \left( \frac{1}{2} Z \star q Z + \frac{1}{3} Z \star Z \star Z \right) - \frac{1}{4} \int_{\partial \mathcal{M}_9} \text{Tr}_{\mathcal{A}} [h \pi_h(Z) \star Z] \ , \quad (2.28)$$

where  $\pi_h$  is the automorphism sending  $h$  to  $-h$ , and

$$q := h d \ . \quad (2.29)$$

Keeping in mind that  $f \in \mathcal{E}$ , the operation  $\text{Tr}_{\mathcal{A}}$  is defined as

$$S = \int_{\mathcal{M}_9} \text{Tr}_{\mathcal{A}} f := \int_{\mathcal{X}_5} \text{STr}_{\Omega(\mathcal{Z}_4)} \text{Tr}_{\mathcal{F}} \text{Tr}_{\mathcal{W}_0} \text{Tr}_{\mathcal{K}} f \ , \quad (2.30)$$

where  $\text{STr}_{\Omega(\mathcal{Z}_4)}$  is defined in (2.6), and the remaining trace operations are defined as

$$\text{Tr}_{\mathcal{F}} \sum_{i,j} e_{ij} M^{ij}(h) = M^{11}(0) + M^{22}(0) , \quad (2.31)$$

$$\text{Tr}_{\mathcal{W}_0} P = P^{1,1}(0,0) , \quad (2.32)$$

$$\text{Tr}_{\mathcal{K}} f = f|_{k=0=\bar{k}} , \quad (2.33)$$

with  $P \in \mathcal{W}_0$  from (2.24) which furnishes the definition of  $P^{1,1}(0,0)$ . Defining  $\text{Tr}_{\mathcal{E}} f := \int_{\mathcal{M}_9} \text{Tr}_{\mathcal{A}} f$ , it can be shown that  $\text{Tr}_{\mathcal{E}} f \star g = \text{Tr}_{\mathcal{E}} g \star f$  [3].

The total variation of the FCS action gives

$$\delta S = \int_{\mathcal{M}_9} \text{Tr}_{\mathcal{A}} \delta Z \star R + \frac{1}{2} \oint_{\mathcal{M}_9} \text{Tr}_{\mathcal{A}} h \delta Z \star (Z + \pi_h(Z)) , \quad (2.34)$$

where the Cartan curvature

$$R := qZ + Z \star Z . \quad (2.35)$$

Thus, imposing the boundary condition

$$(Z + \pi_h(Z))|_{\partial\mathcal{M}_9} = 0 , \quad (2.36)$$

one has the equation of motion  $R = 0$ . This equation is Cartan integrable, hence gauge invariant, with transformations

$$\delta Z = q\theta + [Z, \theta]_{\star} , \quad \delta R = [R, \theta]_{\star} . \quad (2.37)$$

One the other hand, the requirement of gauge invariance of the action gives the following boundary conditions

$$(\theta - \pi_h(\theta))|_{\partial\mathcal{M}_9} = 0 . \quad (2.38)$$

In obtaining this result, the property (2.7) plays an important role. Setting aside nontrivial flat connections due to the noncommutativity of the base manifold,  $Z$  can be given on-shell in terms of a gauge function  $L$  (which contains forms in different degrees) and a zero-form integration constant  $C$ , viz.

$$Z = L^{\star(-1)} \star (q + C) \star L , \quad qC = C \star C = 0 , \quad (2.39)$$

where the algebraic condition on  $C$  is a consequence of the fact that the form content is as given in (2.15).

The superconnection  $Z$  is assumed to be globally defined. However, if it is rather given by a set of representatives defined locally on charts that cover  $\mathcal{M}_9$ , the appropriate global definition of the action is described in Ref. [3]. In doing so, it proves convenient to write the action (2.28) in terms of master fields  $(X, P)$  defined via  $Z = hX + P$ . Thanks to the boundary term in (2.28), one finds [3]

$$S = \int_{\mathcal{M}_9} \text{Tr}_{\mathcal{A}} (P \star F^X + \frac{1}{3} P \star P \star P) , \quad (2.40)$$

where  $F^X := dX + hXh \star X$ . The general variation of this action (2.28) reads

$$\delta S = \int_{\mathcal{M}_9} \text{Tr}_{\mathcal{A}} (\delta X \star R^P h + \delta P \star R^X + d(\delta X \star P)) , \quad (2.41)$$



where the total derivatives cancel between neighboring patches (in the interior of  $\mathcal{M}_9$ ) since  $\delta X$  and  $P$  belong to sections. Writing  $R = R^X + R^P$  where

$$R^X := F^X + P \star P, \quad R^P := qP + hX \star P + P \star hX, \quad (2.42)$$

then on shell we have  $R^X = 0$  and  $R^P = 0$ , and we are left with

$$\delta S = \int_{\mathcal{M}_9} \text{Tr}_{\mathcal{A}} d(\delta X \star P) = \oint_{\partial \mathcal{M}_9} \text{Tr}_{\mathcal{A}} \delta X \star P. \quad (2.43)$$

Using crucially the property (2.7) to replace the star product in  $\Omega(\mathcal{Z}_4)$  by the classical product (keeping in mind that  $j_z \star j_z = 0$ ), the variation becomes

$$\delta S = \oint_{\partial \mathcal{M}_9} \text{Tr}_{\mathcal{A}} \delta X \star_{\mathcal{A}} P. \quad (2.44)$$

Hence, if  $X$  is free to fluctuate at  $\partial \mathcal{M}_9$ , it follows from the variational principle that

$$P|_{\partial \mathcal{M}_9} = 0. \quad (2.45)$$

Finally, while action is invariant under the gauge transformations with parameters  $\epsilon^X$ , it transforms into a total derivative under transformations with parameters  $\epsilon^P$ , viz.

$$\delta_{\epsilon^P} S = \int_{\mathcal{M}_9} \text{Tr}_{\mathcal{A}} d(\epsilon^P \star R^X), \quad (2.46)$$

that vanishes provided that  $\epsilon^P$  belongs to the same section as  $P$ , and

$$\epsilon^P|_{\partial \mathcal{M}_9} = 0 \quad (2.47)$$

that is indeed equivalent to (2.38).

Using  $Z = hX + P$ , the gauge transformations (2.37) read

$$\begin{aligned} \delta X &= d\epsilon^X + X \star \epsilon^X - h\epsilon^X h \star X + hPh \star \epsilon^P - \epsilon^P \star P, \\ \delta P &= d\epsilon^P + hXh \star \epsilon^P - h\epsilon^P h \star X + [P, \epsilon^X]_{\star}. \end{aligned} \quad (2.48)$$

## 2.3 Component formulation

The action (2.28), upon using the definitions (2.13), (2.14) and (2.16), takes the form

$$\begin{aligned} S &= \int_{\mathcal{M}_9} \text{Tr}_{\mathcal{W} \otimes \mathcal{K}} \left[ \tilde{U} \star DB + V \star \left( F - B \star \tilde{B} + \frac{1}{3} V^{\star 2} + U \star \tilde{U} \right) \right. \\ &\quad \left. + U \star \tilde{D}\tilde{B} + \tilde{V} \star \left( \tilde{F} - \tilde{B} \star B + \frac{1}{3} \tilde{V}^{\star 2} + \tilde{U} \star U \right) \right], \end{aligned} \quad (2.49)$$

where

$$\begin{aligned} F &:= dA + A \star A, & \tilde{F} &:= d\tilde{A} + \tilde{A} \star \tilde{A}, \\ DB &:= dB + A \star B - B \star \tilde{A}, & \tilde{D}\tilde{B} &:= d\tilde{B} + \tilde{A} \star \tilde{B} - \tilde{B} \star A, \\ DU &:= dU + A \star U - U \star \tilde{A}, & \tilde{D}\tilde{U} &:= d\tilde{U} + \tilde{A} \star \tilde{U} - \tilde{U} \star A, \\ DV &:= dV + A \star V + V \star A, & \tilde{D}\tilde{V} &:= d\tilde{V} + \tilde{A} \star \tilde{V} + \tilde{V} \star \tilde{A}. \end{aligned} \quad (2.50)$$

The bulk equations of motion, which amount to vanishing Cartan curvatures, read

$$\begin{aligned}
F - B \star \tilde{B} + V \star V + U \star \tilde{U} &= 0, & DB + V \star U + U \star \tilde{V} &= 0, \\
\tilde{F} - \tilde{B} \star B + \tilde{V} \star \tilde{V} + \tilde{U} \star U &= 0, & \tilde{D}\tilde{B} + \tilde{V} \star \tilde{U} + \tilde{U} \star V &= 0, \\
DU + B \star \tilde{V} + V \star B &= 0, & DV + B \star \tilde{U} - U \star \tilde{B} &= 0, \\
\tilde{D}\tilde{U} + \tilde{B} \star V + \tilde{V} \star \tilde{B} &= 0, & \tilde{D}\tilde{V} + \tilde{B} \star U - \tilde{U} \star B &= 0.
\end{aligned} \tag{2.51}$$

The gauge parameter can be written as  $\theta = \epsilon^X + h\epsilon^P$ , where

$$\epsilon^X = \begin{pmatrix} \epsilon & \eta \\ \tilde{\eta} & \tilde{\epsilon} \end{pmatrix}, \quad \epsilon^P = \begin{pmatrix} \epsilon^V & \eta^U \\ \eta^{\tilde{U}} & \epsilon^{\tilde{V}} \end{pmatrix}. \tag{2.52}$$

The transformation rules for the component fields can be readily obtained from (2.48) by using the definitions (2.13) and (2.14). Thus, on  $\partial\mathcal{M}_9$ , where  $(U, \tilde{U}; V, \tilde{V})$  vanish, one finds

$$\begin{aligned}
F - B \star \tilde{B} &= 0, & DB &= 0, \\
\tilde{F} - \tilde{B} \star B &= 0, & \tilde{D}\tilde{B} &= 0,
\end{aligned} \tag{2.53}$$

which is the desired modification of Vasiliev's original system [15, 16]. Alternatively, going to the basis

$$\tilde{A} = W + K, \quad A = W - K, \tag{2.54}$$

the equations of motion on  $\partial\mathcal{M}_9$  read

$$\begin{aligned}
F_W + K \star K - \frac{1}{2}\{B, \tilde{B}\}_\star &= 0, & D_W K - \frac{1}{2}[\tilde{B}, B]_\star &= 0, \\
D_W B - \{K, B\}_\star &= 0, & D_W \tilde{B} + \{K, \tilde{B}\}_\star &= 0,
\end{aligned} \tag{2.55}$$

where we have defined  $D_W f = df + W \star f - (-1)^{\deg(f)} f \star W$  and  $F_W = dW + W^2$ . Since  $\epsilon^P|_{\partial\mathcal{M}_9} = 0$ , recalling the notation (2.52), and splitting the gauge parameters  $(\epsilon, \tilde{\epsilon})$  as

$$\epsilon = \alpha - \beta, \quad \tilde{\epsilon} = \alpha + \beta, \tag{2.56}$$

the gauge transformations under which the field equations (2.55) are invariant can be written as

$$\begin{aligned}
\delta W &= D_W \alpha + [K, \beta]_\star + \frac{1}{2}\{\tilde{\eta}, B\}_\star + \frac{1}{2}\{\tilde{B}, \eta\}_\star, \\
\delta K &= D_W \beta + [K, \alpha]_\star + \frac{1}{2}[\tilde{\eta}, B]_\star + \frac{1}{2}[\tilde{B}, \eta]_\star, \\
\delta B &= D_W \eta + [B, \alpha]_\star - [K, \eta]_\star + \{B, \beta\}_\star, \\
\delta \tilde{B} &= D_W \tilde{\eta} + [\tilde{B}, \alpha]_\star + [K, \tilde{\eta}]_\star - \{\tilde{B}, \beta\}_\star.
\end{aligned} \tag{2.57}$$

## 2.4 Comparison with the duality extended Vasiliev system

In Ref. [3] it has been shown that by taking  $K = 0$  and choosing  $\tilde{B}$  appropriately, the equations of motion on  $\partial\mathcal{M}_9$  given in (2.55) take the form

$$\begin{aligned}
F_W - \mathcal{V} \star J + \bar{\mathcal{V}} \star \bar{J} + \mathcal{U}_0 \star J \star \bar{J} + \mathcal{U}_1 \star J_{[2]} + \mathcal{U}_2 \star J_{[4]} &= 0, \\
D_W B &= 0,
\end{aligned} \tag{2.58}$$

where

$$J := -\frac{i}{8} dz^\alpha dz_\alpha \kappa_z \star \kappa_y \star k \star (1 + k \star \bar{k}) , \quad \bar{J} := J^\dagger , \quad (2.59)$$

with  $j_z$  defined in (2.8). Furthermore,  $J_{[2]}$  and  $J_{[4]}$  are forms that belong to the de Rham cohomology on  $\mathcal{X}_4$ , and  $(\mathcal{V}, \bar{\mathcal{V}}, U_0, U_1, U_2)$  are star polynomial function of the form  $f(B) = \sum_{n=0}^{\infty} f_n B^{*(n+1)}$ .

It is interesting to compare this system with Vasiliev's recently proposed extended system [49], adapted to our notation, given by

$$\begin{aligned} F_W - \mathcal{V} \star J + \bar{\mathcal{V}} \star \bar{J} + \mathcal{U}_0 \star J \star \bar{J} + g J \star \bar{J} + \mathcal{L}_{[2]} + \mathcal{L}_{[4]} &= 0 , \\ D_W B &= 0 , \end{aligned} \quad (2.60)$$

where  $\mathcal{L}_{[2]}$  and  $\mathcal{L}_{[4]}$  are two new dynamical fields, referred to as Lagrangian forms, given by globally defined central and closed elements of degrees two and four, respectively. As far as the local dynamics is concerned, the two systems are equivalent in form degrees zero and one, since one can always choose a representative for  $\mathcal{L}_{[2]}$  that vanishes in a given coordinate chart. In higher form degrees, the duality extended Vasiliev system contains the term  $g J \star \bar{J}$  and the Lagrangian forms, which are not present in the FCS system<sup>2</sup>. In Ref. [49], the integral  $\oint \mathcal{L}_{[2]}$  has been interpreted as a black hole charge, as has been substantiated black hole solution [50]. As for the integral of  $\mathcal{L}_{[4]}$  over spacetime, it has been proposed [49] as the generating functional of correlators within the context of holography<sup>3</sup>. An important open problem in this framework is how to account for loop corrections. It has been suggested that the quantum mechanical effects may emerge from classical dynamics in an infinite dimensional space that has enough room to describe all multiparticle states in the system [51]. If true, this would be a drastically new way of looking at quantum gravity. The tests of these proposals remain to be seen.

In the approach of Ref. [3] a path integral formulation of the FCS model is proposed along the same lines as the AKSZ construction of Ref. [35] within the geometric framework of Ref. [5]. In this approach, the terms proportional to the closed and central elements in (2.58), which are similar to the Lagrangian form terms in (2.60) but play a different role, as the computation of the effective action proceeds in this case by means of path integral quantization rules which necessarily involves the FCS action itself. The advantage of this approach is the availability of path integral formulation for quantization. The computation of quantum effects are left to future work but an outline of the the role of certain topological invariants in the construction of the on-shell effective action is given in Ref. [3], which we summarize below.

## 2.5 On-shell actions from topological invariants

Starting from an AKSZ path integral on  $\mathcal{M}_9 = [0, \infty[ \times \mathcal{X}_4 \times \mathcal{Z}_4$ , where all fields vanish at  $\{\infty\} \times \mathcal{X}_4 \times \mathcal{Z}_4$  and in addition  $P|_{\{0\} \times \mathcal{X}_4 \times \mathcal{Z}_4} = 0$ , as required by the Batalin–Vilkovisky master equation, one finds that  $S_H$  vanishes on-shell. Following [5], one may generate an on-shell action by adding to  $S_H$  a globally defined

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<sup>2</sup> Whether such coupling can be obtained either by expanding  $B$  around a constant background value or allowing the dependence of  $\tilde{B}$  on  $B$  to contain a simple pole, remains to be seen.

<sup>3</sup> Another proposal for the black hole entropy and generating functional of correlators in higher spin gravity has been made in Ref. [5].

boundary term  $S_{\text{top}} = \oint_{\partial\mathcal{M}_9} \mathcal{V}(X, dX)$ , whose total variation vanishes off-shell, i.e.  $S_{\text{top}}$  is a topological invariant. by its evaluation [49] on the Didenko–Vasiliev Assuming that  $S_{\text{top}}$  does not affect the boundary condition on  $P$  nor the equations of motion, one may argue that the on-shell action is given by  $S_{\text{top}}$ .

Aspects of topological invariants for a general structure group are discussed in Ref. [3], where it is also shown that taking it to be generated by  $\alpha$ -transformation displayed in (2.57), one has the invariants

$$S_{\text{top}}[W, K] = \sum_{p=0}^2 \sum_{n=1}^{p+2} \oint_{\mathcal{X}_{2p} \times \mathcal{Z}_4} \beta_{n,p} \left( \frac{d}{dt} \right) \text{Tr}_{\mathcal{W}_0 \otimes \mathcal{K}} (F_{W_t})^{\star n} \Big|_{t=0} . \quad (2.61)$$

where  $\mathcal{X}_{2p} \subset \mathcal{X}_4$  are closed subsets of dimension  $2p$  for  $p = 0, 1, 2$ ;

$$W_t = W + tK , \quad F_{W_t} = F_W + tD_W K + t^2 K \star K , \quad (2.62)$$

and  $\beta_{n,p}$  are linear differential operators of order  $(2n - 1)$  in  $d/dt$  with constant coefficients. Thus, there are 2, 3, 4 invariants for  $p = 0, 1, 2$ , respectively. The on-shell value of  $S_{\text{top}}[W, K]$  is built out of integrals of traces of  $B \star \tilde{B}$ ,  $\tilde{B} \star B$  and  $K \star K$  forming a finite set of invariants. The observables are invariant off shell under gauge transformations with parameter  $\alpha$ , and on shell using parameters  $(\beta, \eta, \tilde{\eta})$ . In the semi-classical limit, one has the partition function [3]

$$Z_{\text{FCS}} = \sum_{\text{saddles}} \mathcal{N} e^{iS_{\text{top}}} . \quad (2.63)$$

## 2.6 Linearized Fluctuations

The theory on the boundary of  $\mathcal{M}_9$  admits the vacuum solutions

$$\tilde{B}^{(0)} = I , \quad W^{(0)} = L^{-1} \star dL , \quad K^{(0)} = 0 , \quad B^{(0)} = 0 , \quad (2.64)$$

where  $L$  is a gauge function (consisting of forms) and  $I$  is a closed and central element on  $\partial\mathcal{M}_9$ . In particular, to describe Vasiliev's phase of the theory, it is assumed that

$$I = J_{\mathcal{X}} + e^{i\theta_0} J - e^{-i\theta_0} \bar{J} , \quad (2.65)$$

where  $J_{\mathcal{X}}$  is a closed a central element on  $\mathcal{X}_4$ ,  $J$  and  $\bar{J}$  are the closed and central elements on  $\mathcal{Z}_4$  defined in (2.59), and  $\theta_0$  is an arbitrary real constant. The fluctuations in the boundary fields can be expanded as

$$(W - W^{(0)}, B, K, \tilde{B} - \tilde{B}^{(0)}) = \sum_{n \geq 1} (W^{(n)}, B^{(n)}, K^{(n)}, \tilde{B}^{(n)}) . \quad (2.66)$$

At the first order, the equations of motion (2.55) read

$$\begin{aligned} D^{(0)} W^{(1)} - \frac{1}{2} \{I, B^{(1)}\}_{\star} &= 0 , & D^{(0)} K^{(1)} &= 0 , \\ D^{(0)} B^{(1)} &= 0 , & D^{(0)} \tilde{B}^{(1)} + \{I, K^{(1)}\}_{\star} &= 0 , \end{aligned} \quad (2.67)$$

and the abelian gauge transformations following from (2.57) are given by

$$\delta W^{(1)} = D^{(0)} \alpha^{(1)} + \frac{1}{2} \{I, \eta^{(1)}\}_{\star} , \quad \delta B^{(1)} = D^{(0)} \eta^{(1)} , \quad (2.68)$$

$$\delta K^{(1)} = D^{(0)} \beta^{(1)} , \quad \delta \tilde{B}^{(1)} = D^{(0)} \tilde{\eta}^{(1)} - \{I, \beta^{(1)}\}_\star . \quad (2.69)$$

Expressing the first order fluctuations as

$$(W^{(1)}, B^{(1)}, K^{(1)}, \tilde{B}^{(1)}) = L^{-1} \star (W^{(1)'}, B^{(1)'}, K^{(1)'}, \tilde{B}^{(1)'}) \star L , \quad (2.70)$$

where  $L$  is a gauge function and the primed fields are independent of  $x$ , the linearized equations (2.67) are solved by [3]

$$B^{(1)'} = B_{[0]}^{(1)'} + d\rho_v B^{(1)'} , \quad (2.71)$$

$$W^{(1)'} = d\rho_v W^{(1)'} - \frac{1}{2} (d\rho_v - 1) \left( \{\rho_v I, B_{[0]}^{(1)'}\}_\star + \{I, \rho_v B^{(1)'}\}_\star \right) , \quad (2.72)$$

$$K^{(1)'} = d\rho_v K^{(1)'} , \quad (2.73)$$

$$\tilde{B}^{(1)'} = d\rho_v \tilde{B}^{(1)'} + (d\rho_v - 1) \{I, \rho_v K^{(1)'}\}_\star , \quad (2.74)$$

where  $B_{[0]}^{(1)'}$  the zero-form integration constant which harbours the local degrees of freedom of the system and the homotopy contractor  $\rho_v$ , with the convenient choice of  $v = z^\alpha \partial_\alpha$  is defined by

$$\rho_v f(Z, Y, dZ) = Z^\alpha \frac{\partial}{\partial dZ^\alpha} \int_0^1 dt \frac{1}{t} f(tZ, Y, tdZ) . \quad (2.75)$$

The connection  $W_{[1]}^{(1)}$  consists of a pure gauge solution, as its gauge function and gauge parameter belong to the same spaces, plus a set particular solutions that carrying the aforementioned local massless degrees of freedom.

The fields  $K^{(1)}$  and  $\tilde{B}^{(1)'}$ , on the other hand, may introduce new topological degrees of freedom arising in cohomological spaces given by spaces of gauge functions over the spaces of gauge parameters.

In particular,  $\tilde{B}_{[2]}^{(1)'}$  contains moduli associated to the gauge function  $\rho_v \tilde{B}_{[2]}^{(1)'} = \rho_v (e^{i\theta_0} J - e^{-i\theta_0} \bar{J})$ , as  $d\rho_v (e^{i\theta_0} J - e^{-i\theta_0} \bar{J}) = e^{i\theta_0} J - e^{-i\theta_0} \bar{J}$  belongs to an admissible section for  $\tilde{B}_{[2]}^{(1)'}$  while  $\rho_v (e^{i\theta_0} J - e^{-i\theta_0} \bar{J})$  does not belong to an admissible section for  $\tilde{\eta}^{(1)}$ . In more detail it is shown in Ref. [3] that the moduli of  $\tilde{B}_{[2]}^{(1)'}$  can be associated to modes that blow up at infinity, i.e. at the commutative point of  $\mathcal{Z}_4$ .

Going to more general backgrounds for  $\mathcal{M}_9$ , it follows from the fact that the fields  $K$ ,  $B$  and  $\tilde{B}$  belong to sections of the structure group that they can contain topological degrees of freedom provided that there are matching elements in the de Rham cohomology, whose rôle remains to be investigated further. Likewise, going to higher order in perturbation theory, the moduli of  $\tilde{B}$  will generate interaction terms which are expected to have important consequences in the perturbative expansion of the theory and the computation of the correlation functions.

The above linearization suffices to show that the perturbative degrees of freedom of the system are contained in the initial data for the Weyl zero-form. However, in order obtain Fronsdal field equations one has switch from Weyl order to normal order and perform a change of gauge in order to make direct contact with Vasiliev's original perturbative expansion (in which  $z^\alpha A_\alpha = 0$  in normal order), which complies

with the Central On Mass Shell Theorem (COMST). It is important that despite the fact that the FCS model is formulated in the Weyl order, for reasons explained in Section 4, its physical spectrum agrees with the Vasiliev theory, and hence its perturbative expansion should obey the COMST as well. Although a naive transformation of the perturbatively defined master fields from normal to Weyl order is known to produce singularities [52]<sup>4</sup>, the FCS master fields belong to an extended class of symbols, including inner Klein operators, which yields a well-defined perturbation theory in a specific holomorphic gauge (defined by  $z^\alpha A_\alpha = 0$  in Weyl order). Indeed, working with definite boundary conditions (corresponding to generalized Type D solutions [54]), the resulting linearized fields can be mapped to Vasiliev’s basis. We plan to examine whether this remains the case for more general boundary conditions and to higher orders in the perturbative expansion.

### 3 Fractional spin gravity theory

In  $2 + 1$  dimensions Vasiliev’s higher-spin gravity, or more specifically the Prokushkin–Vasiliev model [8], admits truncation to Chern–Simons higher spin gravity [9, 55]. In Ref. [55] the gauge connection is valued in a higher spin algebra that consists of monomials of Wigner–Heisenberg deformed oscillator operators. Monomials of the same order transform in spinor-tensorial representations of the  $AdS_3$  algebra, with arbitrary half-integer spins (which includes integer and half-integer values), and the correspondent fields have standard boson or fermion statistics. However, in  $2 + 1$  dimensional spacetimes the representations of the  $so(2, 1)$  algebra admit spin interpolating half-integer numbers [56, 57] and are referred to as fractional. The physical realisations of fractional spins are known as anyons, and their statistics interpolates between bosons and fermions [58, 59, 60].

As higher spin gravity aims at describing fields with arbitrary spin, for completeness, in three dimensions it should be extended to incorporate fundamental fractional-spin fields. The first step to achieve this goal was given in [61] using operator formalism. Later this was done in Ref. [2] by means of deformation quantization methods — i.e. using star-products [62].

The model constructed in Ref. [2] is a Chern–Simons theory for a gauge field that can be expressed in the form

$$\mathbb{A} = \begin{bmatrix} W & \psi \\ \bar{\psi} & U \end{bmatrix} \cong \begin{bmatrix} \text{HS gravity} & \text{Fractional spin} \\ \text{Fractional spin} & \text{Internal interactions} \end{bmatrix}, \quad (3.1)$$

where the blocks correspond to four different sectors of the gauge algebra of the theory,  $\mathcal{A}(2; \nu|w)$ , dubbed fractional-spin algebra, which we shall introduce below. To these sectors we associate a higher-spin gravity connection  $W$ , an internal connection 1-form  $U$  and the “fractional-spin gravitinos”  $(\psi, \bar{\psi})$ . Indeed, the Chern–Simons action obtained for (3.1) resembles the Achúcarro–Townsend theory [7] of supergravity in three dimensions. However, any attempt to extend standard supergravity with fractional-spin fields would lead to higher spin gravity. Since fractional-spin Lorentz representations are infinite dimensional, there appear infinitely many additional symmetries that can be gauged: The higher spin symmetries. The naive matrix

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<sup>4</sup>For a general discussion of ordering schemes and maps between them, see *e.g.* [53].

(super)trace gives rise, in this context, to divergences that cannot readily be regularised consistently without changing the definition of super-traces of matrices. One achievement of [2] can be regarded as a solution to this problem; see [63] and [64] for a related discussion and an extension of the naive matrix trace.<sup>5</sup>

In what follows we present the main points that lead to the formulation of fractional-spin gravity as presented in Ref. [2]. Here we give a somewhat simplified presentation compared the more technical work [2] to which we refer for a complete and precise treatment.

### 3.1 The fractional spins algebra

The building block of the fractional spin algebra is the algebra  $Aq(2; \nu)$  introduced by Vasiliev [55] and identified with the universal enveloping algebra of the deformed oscillator algebra [65, 66] (see also the Refs. [67, 68]), in turn presented by

$$[q_\alpha, q_\beta]_\star = 2i(1 + \nu k)\epsilon_{\alpha\beta} , \quad \{k, q_\alpha\}_\star = 0 , \quad k \star k = 1 , \quad (3.2)$$

$$(q_\alpha)^\dagger = q_\alpha , \quad k^\dagger = k , \quad \nu \in \mathbb{R} . \quad (3.3)$$

The associative algebra  $Aq(2; \nu)$  consists of arbitrary star polynomials (of finite degree) in  $(q_\alpha, k)$ , which in Weyl order read

$$T_{\alpha(n)} := q_{\alpha_1} \cdots q_{\alpha_n} \equiv q_{(\alpha_1} \star \cdots \star q_{\alpha_n)} , \quad T_{\alpha(n)} \star k , \quad (3.4)$$

where the symmetrisation has unit strength. We split the algebra in four sectors, using the projected elements

$$T_{\alpha(n)}^{\sigma, \sigma'} := [q_{\alpha_1} \cdots q_{\alpha_n}]^{\sigma, \sigma'} := \Pi^\sigma \star q_{(\alpha_1} \star \cdots \star q_{\alpha_n)} \star \Pi^{\sigma'} , \quad \Pi^\pm = \frac{1}{2}(1 \pm k) , \quad (3.5)$$

which are non-vanishing iff  $\sigma\sigma' = (-1)^n$ . These projections belong to the sub-algebras  $Aq(2; \nu)^{\sigma, \sigma'} = \Pi^\sigma \star Aq(2; \nu) \star \Pi^{\sigma'} \in Aq(2; \nu)$ . Formally, the space  $Aq(2; \nu)$  does not contain distribution (non-polynomial) class of functions. It is necessary, in order to support fractional spins, to extend  $Aq(2; \nu)$  with certain “ $w$ -class distributions”, where  $w$  refers to the operator which is diagonalised by them. We will refer to the algebra of these elements as  $Aw(2; \nu)$ . The operator  $w$  appears in the definition of the spin operator

$$J_0 = \frac{1}{2} w \star \Pi^+ , \quad w := \frac{1}{4} (\tau_0)^{\alpha\beta} q_\alpha \star q_\beta , \quad (3.6)$$

which is here chosen in a non-standard form (cf. [55]), as it involves a projector  $\Pi^+$ .  $J_0$  generates the rotations in the spatial plane, and the projection  $\Pi^+$  plays an essential role in order to create fractional spins in the connection. More generally, the spin part of the Lorentz transformation is generated by

$$J_a = \frac{1}{4} (\tau_a)^{\alpha\beta} J_{\alpha\beta} , \quad J_{\alpha\beta} = \frac{1}{2} q_{(\alpha} \star q_{\beta)} \star \Pi^+ \in Aq(2; \nu)^{+,+} , \quad a = 0, 1, 2 , \quad (3.7)$$

where in terms of Pauli matrices,

$$(\tau_a)^{\alpha\beta} = (\tau_a)^{\beta\alpha} = (\mathbb{1}, \sigma^1, \sigma^3) , \quad (\tau^a)_\alpha{}^\beta = (\tau^a)^{\alpha'\beta} \epsilon_{\alpha'\alpha} = (-i\sigma^2, -\sigma^3, \sigma^1) . \quad (3.8)$$

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<sup>5</sup>We are grateful to A. Campoleoni and T. Procházka for discussions on this issue.

Note that here  $(\tau^a)_\alpha{}^\beta$  generates a real Clifford algebra and that the conjugation matrix  $\epsilon$  rises and lower spinor indices. For more detail about these conventions the reader may consult the Ref. [2].

The distribution needed to describe fractional spin are solutions of the “star-genvalue” problem,

$$2J_0 \star T_E = (w \star \Pi^+) \star T_E = E T_E . \quad (3.9)$$

The operator  $w$  can also be expressed in terms of ladder operators defined as

$$w = a^+ a^- = \frac{1}{2} \{a^-, a^+\}_\star , \quad [w, a^\pm]_\star = \pm a^\pm , \quad (3.10)$$

where

$$a^\pm = u^{\pm\alpha} q_\alpha , \quad u^{+\alpha} u_\alpha^- = -\frac{i}{2} , \quad (u_\alpha^\pm)^\dagger = u_\alpha^\mp . \quad (3.11)$$

Since powers of  $w$  form a closed subalgebra, as we shall see below, in order to solve (3.9) we expand their  $\star$ -genfunctions as<sup>6</sup>

$$T_E = \sum_{m=0}^{\infty} f_m w^m \star \Pi^+ , \quad w^m = (a^+)^m (a^-)^m , \quad (3.12)$$

where  $f_m$  are constants. We can verify that

$$a^\pm \star [w^m]^{\sigma,\sigma} = \left[ a^\pm \left( w^m \mp \frac{m(2m+1-\nu\sigma)}{2(2m+1)} w^{m-1} \right) \right]^{-\sigma,\sigma} , \quad (3.13)$$

and hence

$$w \star [w^m]^{\sigma,\sigma} = [w^{m+1} + \lambda_m^\sigma w^{m-1}]^{\sigma,\sigma} , \quad (3.14)$$

where we have defined

$$\lambda_m^\sigma = -\frac{m^2}{4} \frac{(2m+1-\nu\sigma)(2m-1+\nu\sigma)}{(2m+1)(2m-1)} . \quad (3.15)$$

Using (3.14) in (3.9) yields a recursive formulas. To restrict the space of solutions, and using our intuition on the harmonic oscillator in Fock space, we can solve the lowest weight condition to identify the ground state<sup>7</sup>

$$a^- \star T_{E_0} = 0 , \quad (3.16)$$

for which we find a unique solution with  $f_0 = 1$ , given by

$$f_m = \frac{(-2)^m}{m!} \frac{\left(\frac{3}{2}\right)_m}{\left(\frac{3-\nu}{2}\right)_m} , \quad (3.17)$$

where the Pochhammer symbol  $(a)_n$  is given by 1 if  $n = 0$  and by  $a(a+1)\cdots(a+n-1)$  if  $n = 1, 2, \dots$ . Hence, using the definition of the confluent hypergeometric function, viz.

$${}_1F_1(a; b; z) = \sum_{n \geq 0} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} , \quad (3.18)$$

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<sup>6</sup>Notice that this amount to taking  $\sigma = +1$  in the conventions of Ref. [2].

<sup>7</sup>Here we are taking the choice  $\epsilon = +1$  in the notation of [2].



we have

$$T_{E_0} = {}_1F_1\left(\frac{3}{2}; \frac{3-\nu}{2}; -2w\right) \star \Pi^\sigma, \quad (3.19)$$

which obeys

$$(w - E_0) \star T_{E_0} = 0, \quad E_0 = \frac{1+\nu}{2}, \quad (3.20)$$

by virtue of (3.14).  $T_{E_0}$  can be identified with the ground state non-normalised projector  $|0\rangle\langle 0|$  of the *deformed harmonic oscillator* introduced by Wigner [65, 66]. Higher states can be generated by the left and right  $\star$ -multiplication of elements of  $Aq(2; \nu)$ , in terms of  $\star$ -powers of ladder operators,

$$|m\rangle\langle n| = (a^+)^{\star n} \star |0\rangle\langle 0| \star (a^-)^{\star n}, \quad |0\rangle\langle 0| := T_{E_0}, \quad (3.21)$$

such that

$$(w - E_m) \star |m\rangle\langle n| = 0 = |m\rangle\langle n| \star (w - E_n), \quad E_m = \left(m + \frac{1+\nu}{2}\right). \quad (3.22)$$

The projector  $\Pi^+$  in the definition of  $J_a$  makes its action trivial on odd parity labels since  $\Pi^+ \star |2n+1\rangle\langle m| = |m\rangle\langle 2n+1| \star \Pi^+ = 0$ .

Introducing the projections of the algebra  $Aw(2; \nu)$

$$Aw(2; \nu)^{\sigma, \sigma'} = \Pi^\sigma \star Aw(2; \nu) \star \Pi^{\sigma'}, \quad (3.23)$$

the fractional spin algebra is defined by specification of four sectors

$$\mathcal{A}(2; \nu|w) := \begin{bmatrix} Aq(2; \nu)^{+,+} & Aw(2; \nu)^{+,-} \\ Aw(2; \nu)^{-,+} & Aw(2; \nu)^{-,-} \end{bmatrix} \quad \ni \quad \mathbb{A} = \begin{bmatrix} W & \psi \\ \bar{\psi} & U \end{bmatrix}, \quad (3.24)$$

to which the gauge connection (3.1) belongs to.

Thus, in each sector the gauge connection must be expanded as follows,

$$\begin{aligned} \mathbb{A} &= \begin{bmatrix} W & \psi \\ \bar{\psi} & U \end{bmatrix} \\ &= \begin{bmatrix} W = \sum_n W^{\alpha(n)} q_{\alpha(n)} \star \Pi^+ & \psi = \sum_{m,n \geq 0} \psi^{mn} |2m\rangle\langle 2n+1| \\ \bar{\psi} = \sum_{n,m \geq 0} \bar{\psi}^{mn} |2m+1\rangle\langle 2n| & U = \sum_{n,m \geq 0} U^{mn} |2m+1\rangle\langle 2n+1| \end{bmatrix}. \end{aligned} \quad (3.25)$$

These sectors satisfy the “fusion rules”

$$\mathcal{A}(2; \nu|w)^{\sigma, \sigma'} \star \mathcal{A}(2; \nu|w)^{\tau, \tau'} \cong \delta^{\sigma', \tau} \mathcal{A}(2; \nu|w)^{\sigma, \tau'}, \quad (3.26)$$

where the projections

$$\mathcal{A}(2; \nu|w)^{\sigma, \sigma'} = \Pi^\sigma \star \mathcal{A}(2; \nu|w) \star \Pi^{\sigma'}, \quad (3.27)$$

fall in the correspondent blocks of (3.24).

Let us perform now a (global) Lorentz transformation

$$\mathbb{A}' = g_e \star \mathbb{A} \star g_e^{-1}, \quad g_e = \exp_\star(i\epsilon^a J_a). \quad (3.28)$$

Because the projector  $\Pi^+$  in its definition,  $J_a$  acts non-trivially only in the upper diagonal block (of higher spin gravity) and on the off-diagonal blocks (“fractional spin gravitinos”), while the lower diagonal block  $U$  (of internal interactions) does not transform. Internal transformations, whose basis of generators is given by  $|2m+1)(2n+1|$  (up to normalisations and reality conditions), act non-trivially on fractional spin fields and on itself.

As we are interested to show how fractional spins make their appearance, let us perform a rotation by  $2\pi$ , focusing in the sector  $\mathcal{A}(2; \nu|w)^{\pm, \mp}$ . The parameter of transformation is given by  $\epsilon = 2\pi J_0$ , hence single elements of the basis of the sectors  $\mathcal{A}(2; \nu|w)^{\pm, \mp}$  transform as

$$g_{2\pi} \star |2m)(2n+1| \star g_{-2\pi} = e^{i\pi(m+\frac{1+\nu}{4})} |2m)(2n+1|, \quad (3.29)$$

$$g_{2\pi} \star |2n+1)(2m| \star g_{-2\pi} = e^{-i\pi(m+\frac{1+\nu}{4})} |2n+1)(2m|, \quad (3.30)$$

hence it follows that the projector basis  $|\text{odd})(\text{even}| \oplus |\text{even})(\text{odd}|$  have non-fermionic/bosonic statistical phases. Note that for odd values of  $\nu$  the phases become fermionic or bosonic. The cases  $\nu$  negative-odd are critical, in the sense that the representations of the Lorentz algebra become non-unitary and decouple in two sectors, of finite dimension and non-unitary, and of infinite dimension and unitary. For positive-odd  $\nu$  the representations of half-integer spin of the Lorentz algebra are unitary and infinite dimensional. Thus, for half-integer values of the spin, there appear different representations of the Lorentz group. Indeed, the second (infinite dimensional case) is more exotic, since the field theory of these type of representation is less known, while in the finite dimensional case the field theories are standard, including e.g. Dirac, Rarita–Schwinger, or Bargmann–Wigner, and well known gauge gravity models including  $SL(N)$ -like Chern–Simons higher spin gravities in three dimensions. For field theories of infinite dimensional representations with half-integer spins the reader can consult the Refs. [69, 70, 71, 72]. More details and complete analysis on the fractional spin algebra and its critical limits can be found in Refs. [61, 2].

### 3.2 Gravitational and gauge couplings from Chern-Simons fractional spin gravity

For a polynomial class of functions, with elements  $f(q; k)$  Vasiliev’s super-trace is given by,

$$\text{STr}_{Aq(2; \nu)} f(q; k) = f(0; -\nu) \quad (3.31)$$

It is not straightforward that Vasiliev’s super-trace (3.31) will be consistent when operating on elements of the fractional spin algebra, because the presence of non-polynomial functions. It suffices to establish two consistency conditions: (i) Finite star products, ii) Finite Vasiliev supertraces (which together with (i) implies cyclicity).

The condition (i) implies that

$$|0)(0| \star |0)(0| = \mathcal{N}^{-1} |0)(0|, \quad \mathcal{N} \in ]0, \infty[, \quad (3.32)$$

where  $\mathcal{N}$  is a normalisation constant. This calculation was verified up to first order in  $w$ -power series and at all order in  $\nu$ . Assuming that this remains true for all order in  $w$  implies the existence of a normalised

projector

$$|0\rangle\langle 0| := \mathcal{N} |0\rangle\langle 0| , \quad |0\rangle\langle 0| \star |0\rangle\langle 0| = |0\rangle\langle 0| , \quad (3.33)$$

and the normalisation can be obtained trusting on the supertrace applied to the latter product, for which we need to compute just the zero order term in the  $w$ -expansion. Doing so we obtain

$$\mathcal{N}^{-1} = \frac{1-\nu}{2} . \quad (3.34)$$

Before writing down the Chern–Simon action, we should introduce a Clifford algebra  $\{\mathbb{1}, \Gamma\}$ ,  $\Gamma^2 = \mathbb{1}$ , which will allow us to double<sup>8</sup> the fractional higher spin algebra and embed in it the  $AdS_3$  isometry algebra  $so(2,2) \cong so(2,1) \oplus so(2,1)$ . The supertrace of functions in oscillator variables and  $\Gamma$  is now defined as

$$STr f(q; k; \Gamma) = f(0; -\nu; 0) . \quad (3.35)$$

It can be verified that

$$STr (J_a \star J_b) = \frac{1}{32} (1-\nu^2) (1 - \frac{\nu}{3}) \eta_{ab} , \quad (3.36)$$

and

$$STr (P_m{}^n \star P_{m'}{}^{n'}) = -\delta_m^{n'} \delta_{m'}^n , \quad P_m{}^n := |m\rangle\langle n| \in u(\infty) . \quad (3.37)$$

Comparing with normalised trace operations operations, such that

$$Tr_{\text{grav}}(J_a \star J_b) = \frac{1}{2} \eta_{ab} , \quad Tr_{\text{int}}(P_m{}^n \star P_{m'}{}^{n'}) = \frac{1}{2} \delta_m^{n'} \delta_{m'}^n , \quad (3.38)$$

it follows that

$$STr|_{\text{grav}} = \frac{1}{16} (1-\nu^2) (1 - \frac{\nu}{3}) Tr_{\text{grav}} , \quad (3.39)$$

$$STr|_{\text{int}} = -2 Tr_{\text{int}} , \quad (3.40)$$

where  $STr|_{\cdot}$  means restriction of the supertrace of the fractional spin gravity (3.35) either to the gravity sector or the internal sector respectively. Hence the Chern–Simon action for the fractional spin theory reads

$$S[\mathbb{A}] = \frac{\varkappa}{2\pi} \int_{M_3} Tr_{\mathcal{A}} \left( \frac{1}{2} \mathbb{A} \star d\mathbb{A} + \frac{1}{3} \mathbb{A} \star \mathbb{A} \star \mathbb{A} \right) , \quad (3.41)$$

where the trace  $Tr_{\mathcal{A}}$  of  $\mathbb{F} \in \mathcal{A}(2; \nu|w) \otimes \text{Cliff}(\Gamma)$  is defined by<sup>9</sup>

$$Tr_{\mathcal{A}}[\mathbb{F}] := \text{Str}[\Gamma \star (\mathbb{F}^{+,+} + \mathbb{F}^{-,-})] . \quad (3.42)$$

With the decomposition

$$W(q, k, \Gamma) = \frac{1+\Gamma}{2} \star W_L(q, k) + \frac{1-\Gamma}{2} \star W_R(q, k) , \quad idem \quad U , \quad (3.43)$$

---

<sup>8</sup>The generator  $\Gamma$  was denoted  $\gamma$  in Ref. [2].

<sup>9</sup>Out of the two possibilities for the fractional-spin algebra denoted by  $\mathcal{A}_{\pm}$  in Ref. [2], here we choose  $\mathcal{A}_+$  that we call  $\mathcal{A}$  for short. A proper assignment of semi-classical statistics for the components of the fractional-spin fields  $\psi$  and  $\bar{\psi}$  requires the introduction of a fermionic Klein operator denoted  $\xi$  in Ref. [2] s.t. the components of the one-form  $\mathbb{A} \in \mathcal{A}(2; \nu|w) \otimes \text{Cliff}(\Gamma) \otimes \text{Cliff}(\xi)$  are all bosonic.

the action (3.41) produces

$$S[\mathbb{A}] = \frac{1}{2} S[\mathbb{A}_L] - \frac{1}{2} S[\mathbb{A}_R] , \quad (3.44)$$

where ( $c = L, R$ )

$$S[\mathbb{A}_c] = \frac{\varkappa}{2\pi} \int \left[ \mathcal{L}_{\text{CS}}(W_c) + \mathcal{L}_{\text{CS}}(U_c) + \frac{1}{2} \text{Str} \left( \psi_c \star D\bar{\psi}_c + \bar{\psi}_c \star D\psi_c \right) \right] , \quad (3.45)$$

in terms of the Chern–Simons Lagrangian

$$\mathcal{L}_{\text{CS}}(W) = \text{STr} \left[ \frac{1}{2} W \star dW + \frac{1}{3} W \star W \star W \right] , \quad (3.46)$$

idem  $\mathcal{L}_{\text{CS}}(U)$ , and the covariant derivatives

$$D\psi = d\psi + W \star \psi + \psi \star U , \quad D\bar{\psi} = d\bar{\psi} + U \star \bar{\psi} + \bar{\psi} \star W . \quad (3.47)$$

By comparison with the sum of the standard gravity action and gauge interactions in absence of fractional spin gravitinos and higher spin interactions,

$$\begin{aligned} S_{\text{grav}}[W] + S_{\text{int}}[U] &= \frac{k_{\text{grav}}}{2\pi} \int_{M_3} \text{Tr}_{\text{grav}} \left[ \frac{1}{2} W_{\text{grav}} \star dW_{\text{grav}} + \frac{1}{3} W_{\text{grav}}^{\star 3} \right] \\ &+ \frac{k_{\text{int}}}{2\pi} \int_{M_3} \text{Tr}_{\text{int}} \left[ \frac{1}{2} U \star dU + \frac{1}{3} U^{\star 3} \right] , \end{aligned} \quad (3.48)$$

that gives, up to boundary terms in the gravitational sector, the sum of the Einstein–Hilbert action and an internal Chern–Simons theory for the group  $U(\infty) \otimes U(\infty)$ ,

$$S_{\text{grav}}[W] + S_{\text{int}}[U] = \frac{k_{\text{hs}}}{4\pi\ell} \int d^3x \sqrt{-g} \left( R + \frac{2}{\ell^2} \right) + \frac{k_{\text{int}}}{4\pi} \int_{M_3} \text{Tr}_{\text{int}} \left( U \star dU + \frac{2}{3} U^{\star 3} \right) , \quad (3.49)$$

we find that the higher spin gravity and the internal couplings are given by<sup>10</sup>

$$k_{\text{hs}} = \frac{\varkappa}{32} (1 - \nu^2) \left( 1 - \frac{\nu}{3} \right) , \quad k_{\text{int}} = -\varkappa . \quad (3.50)$$

The Newton constant is given by  $G_N = \ell/(4k_{\text{hs}})$ . The relation between the coupling constants of the (fractional-spin) gravity sector and the internal interaction sector is therefore given by

$$k_{\text{hs}} = -\frac{1}{32} (1 - \nu^2) \left( 1 - \frac{\nu}{3} \right) k_{\text{int}} . \quad (3.51)$$

The interactions predicted by the model (3.41) can be read from the resulting equations of motion:

$$d\mathbb{A} + \mathbb{A} \star \mathbb{A} = 0 , \quad (3.52)$$

which in components are given by

$$dW + W \star W + \psi \star \bar{\psi} = 0 , \quad dU + U \star U + \bar{\psi} \star \psi = 0 , \quad (3.53)$$

---

<sup>10</sup>We take the opportunity to correct a typo appearing in the expression for  $k_{\text{hs}}$  given in the first equation (4.30) of Ref. [2].

$$d\psi + W \star \psi + \psi \star U = 0 \ , \quad d\bar{\psi} + U \star \bar{\psi} + \bar{\psi} \star W = 0 \ . \quad (3.54)$$

Here we observe how the fractional-spin fields source the field strength of higher-spin gravity and the internal interactions, while they couple minimally to the latter interactions, either from the left or right actions. We can visualise this result as saying that fractional-spin charges carry higher-spin gravity fluxes.

To conclude this section, we would like to mention that we have omitted many details for the sake of simplicity. The complete treatment can be found in Ref. [2]. In terms of the notation of Ref. [2], here we took the choice  $\sigma = +1$ ,  $\epsilon = +1$  and considered the fractional-spin algebra  $\mathcal{A}_+$  for which the component fields in the “gravitino” sector are fermionic and multiplied by the fermionic Kleinian  $\xi$ .

## 4 Matter-coupled 3D higher-spin gravity

In this section, we review the results of Ref. [4] where an action was provided for matter-coupled 3D higher-spin gravity, that reproduces upon variation the full nonlinear bosonic Prokushkin–Vasiliev (PV) equations. We take the opportunity to review the PV equations and spell out the truncation of the PV spectrum of matter fields to a single *real* scalar field. This minimal truncation can be useful in the context of the Gaberdiel–Gopakumar conjecture [11]. See Ref. [12] for a review.

### 4.1 Geometric Formulation of Prokushkin–Vasiliev’s system

In this subsection we are going to present a geometric formulation of Prokushkin–Vasiliev systems [8], describing matter coupled to gauge fields in three dimensional spacetime. Master fields consist of a one form  $A$  and a zero form  $B$ , defined on a so-called correspondence space  $\mathcal{M}_5 = \mathcal{M}_3 \times \mathcal{Z}_2$ , where  $\mathcal{M}_3$  is the three dimensional spacetime manifold with local coordinates  $x^\mu$ , and  $\mathcal{Z}_2$  is a non-commutative manifold with coordinates  $z^\alpha$ ,  $\alpha = 1, 2$ . The fields take value in the higher spin algebra that extends  $sp(2, \mathbb{R})$ , generated by twistor variables  $y^\alpha$ , and are tensored with elements  $\Gamma_i$  that generate a Clifford algebra:  $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$ , for  $i, j = 1, \dots, N$ .

$$A = dx^\mu U_\mu(x, z|y; \Gamma_i) + dz^\alpha V_\alpha(x, z|y; \Gamma_i) \ , \quad B = B(x, z|y; \Gamma_i) \ . \quad (4.1)$$

The dependence of the master fields on  $(y^\alpha, z^\alpha)$  is treated using symbol calculus, whereby they belong to classes of functions (or distributions) on  $\mathcal{Y}_2 \times \mathcal{Z}_2$  that can be composed using two associative products: the standard commutative product rule, denoted by juxtaposition, and an additional noncommutative product rule, denoted by a  $\star$ . In what follows, we shall use the normal ordered basis in which the star product rule is defined formally by

$$(f \star g)(y, z) := \int_{\mathbb{R}^4} \frac{d^2 u d^2 v}{(2\pi)^2} e^{iv^\alpha u_\alpha} f(y + u, z + u) g(y + v, z - v) \ , \quad (4.2)$$

whereas a more rigorous definition requires a set of fusion rules. In particular, the above composition rule rigorously defines the associative Weyl algebra  $\text{Aq}(4)$ . This algebra consists of arbitrary polynomials in  $y^\alpha$

and  $z^\alpha$ , modulo

$$y_\alpha \star y_\beta = y_\alpha y_\beta + i\epsilon_{\alpha\beta} , \quad y_\alpha \star z_\beta = y_\alpha z_\beta - i\epsilon_{\alpha\beta} , \quad (4.3)$$

$$z_\alpha \star y_\beta = z_\alpha y_\beta + i\epsilon_{\alpha\beta} , \quad z_\alpha \star z_\beta = z_\alpha z_\beta - i\epsilon_{\alpha\beta} , \quad (4.4)$$

whose symmetric and anti-symmetric parts, respectively, define the normal order and the (ordering independent) commutation rules, viz.<sup>11</sup>

$$[y^\alpha, y^\beta]_\star = -[z^\alpha, z^\beta]_\star = 2i\epsilon^{\alpha\beta} , \quad [y^\alpha, z^\beta]_\star = 0 . \quad (4.5)$$

The basis one-forms  $(dx^\mu, dz^\alpha)$  obey

$$[dx^\mu, f]_\star = 0 = [dz^\alpha, f]_\star , \quad (4.6)$$

where the graded star commutator of differential forms is given by

$$[f, g]_\star = f \star g - (-1)^{\deg(f)\deg(g)} g \star f , \quad (4.7)$$

with  $\deg$  denoting the total form degree on  $\mathcal{M}_3 \times \mathcal{Z}_2$ . To describe bosonic models, we impose

$$\pi(A) = A , \quad \pi(B) = B \quad (4.8)$$

where  $\pi$  is the automorphism of the differential star product algebra defined by

$$\pi(x^\mu, dx^\mu, z^\alpha, dz^\alpha, y^\alpha, \Gamma_i) = (x^\mu, dx^\mu, -z^\alpha, -dz^\alpha, -y^\alpha, \Gamma_i) . \quad (4.9)$$

The hermitian conjugation is defined by

$$(f \star g)^\dagger = (-1)^{\deg(f)\deg(g)} g^\dagger \star f^\dagger , \quad (z_\alpha, dz^\alpha; y_\alpha, \Gamma_i)^\dagger = (-z_\alpha, -dz^\alpha; y_\alpha, \Gamma_i) . \quad (4.10)$$

and the reality conditions on the master fields read

$$A^\dagger = -A , \quad B^\dagger = B . \quad (4.11)$$

Defining

$$F = dA + A \star A , \quad DB = dB + A \star B - B \star A , \quad d = dx^\mu \partial_\mu + dz^\alpha \frac{\partial}{\partial z^\alpha} , \quad (4.12)$$

where the differential obeys

$$d(f \star g) = (df) \star g + (-1)^{\deg(f)} f \star dg , \quad (df)^\dagger = d(f^\dagger) , \quad (4.13)$$

the PV field equations can be written as

$$F + B \star J = 0 , \quad DB = 0 , \quad (4.14)$$

---

<sup>11</sup>The doublet variables  $y^\alpha$  and  $z^\alpha$  form Majorana spinors once the equations are cast into a manifestly Lorentz covariant form.

where

$$J := -\frac{i}{4} dz^\alpha dz_\alpha \kappa \quad \kappa := e^{iy^\alpha z_\alpha} . \quad (4.15)$$

The element  $J$  is closed and central in the space of  $\pi$ -invariant forms, viz.

$$dJ = 0 , \quad J \star f = \pi(f) \star J , \quad (4.16)$$

as can be seen from the fact that  $\kappa$ , which is referred to as the inner Klein operator, obeys

$$\kappa \star f(x, dx, z, dz, y, \Gamma_i) \star \kappa = f(x, dx, -z, dz, -y, \Gamma_i) . \quad (4.17)$$

It follows that (4.14) defines a universally Cartan integrable system (i.e. a set of generalized curvature constraints compatible with  $d^2 \equiv 0$  in any dimension). The Cartan gauge transformations take the form

$$\delta_\epsilon A = d\epsilon + [A, \epsilon]_\star , \quad \delta_\epsilon B = [B, \epsilon]_\star . \quad (4.18)$$

In order to see the equivalence with Prokushkin–Vasiliev systems, let us introduce the oscillator-like fields  $S_\alpha := z_\alpha - 2i V_\alpha$  and split the field equations in  $dx$  and  $dz$  directions, thus obtaining

$$\begin{aligned} d_X U + U \star U &= 0 , \quad d_X B + [U, B]_\star = 0 , \quad d_X S_\alpha + [U, S_\alpha]_\star = 0 , \\ [S_\alpha, B]_\star &= 0 , \quad [S_\alpha, S_\beta]_\star = -2i\epsilon_{\alpha\beta} (1 - B \star \kappa) , \end{aligned} \quad (4.19)$$

where  $d_X := dx^\mu \partial_\mu$  is the spacetime differential. We stress that, due to the bosonic projection, one has  $\{S_\alpha, \kappa\}_\star = 0$ , that will be crucial for the discussion of massive vacua.

**Massless vacua.** Let us analyse the above system around the vacuum solution  $B_0 = 0$ . From the vacuum equation  $[S_{0\alpha}, S_{0\beta}]_\star = -2i\epsilon_{\alpha\beta}$  one can take  $S_{0\alpha} = z_\alpha$  and hence

$$0 = [z_\alpha, U_0]_\star = -2i \frac{\partial U_0}{\partial z_\alpha} \rightarrow U_0 = \Omega(x|y; \Gamma_i) , \quad (4.20)$$

and the remaining equation is the flatness condition  $d\Omega + \Omega \star \Omega = 0$ . Bilinears in  $y$  variables generate  $sp(2, \mathbb{R})$  under star-commutators:

$$T_{\alpha\beta} := \frac{1}{2i} y_\alpha y_\beta , \quad [T_{\alpha\beta}, T_{\gamma\delta}]_\star = 4\epsilon_{(\alpha(\gamma} T_{\beta)\delta)} , \quad (4.21)$$

but some outer element is needed in order to double  $sp(2, \mathbb{R})$  and thus represent the  $AdS_3$  isometry algebra  $sp(2, \mathbb{R}) \oplus sp(2, \mathbb{R})$ . In order to describe massless vacua, it turns out that the minimal dimension of the Clifford algebra is  $N = 2$ , and one can write the vacuum spacetime connection as

$$\Omega(x|y; \Gamma_i) = \frac{1}{4i} \left( \omega^{\alpha\beta}(x) y_\alpha y_\beta + \Gamma_1 e^{\alpha\beta}(x) y_\alpha y_\beta \right) \equiv \omega + \Gamma_1 e , \quad (4.22)$$

where  $\omega^{\alpha\beta}$  and  $e^{\alpha\beta}$  are the background Lorentz connection and dreibein, respectively. The flatness condition amounts then to

$$de_{\alpha\beta} + 2\omega_{(\alpha}{}^\gamma \wedge e_{\beta)\gamma} = 0 , \quad d\omega_{\alpha\beta} + \omega_\alpha{}^\gamma \wedge \omega_{\beta\gamma} = -e_\alpha{}^\gamma \wedge e_{\beta\gamma} , \quad (4.23)$$

that indeed describes  $AdS_3$  spacetime with unit radius. In order to study fluctuations around this vacuum we expand the master fields as  $U = \Omega + w_1 + \dots$ ,  $B = 0 + B_1 + \dots$  and  $S_\alpha = z_\alpha + S_{1\alpha} + \dots$ , and the field equations for first order fluctuations read

$$\begin{aligned} D_0 w_1 &= 0, \quad D_0 B_1 = 0, \quad D_0 S_\alpha + 2i \frac{\partial w_1}{\partial z^\alpha} = 0, \\ \frac{\partial B_1}{\partial z^\alpha} &= 0, \quad \frac{\partial S_1^\alpha}{\partial z^\alpha} = B_1 \star \kappa, \end{aligned} \quad (4.24)$$

from which we see that  $B_1 = C(x|y; \Gamma_i)$  is  $z$ -independent, and the background covariant derivative is defined by  $D_0 f := df + [\Omega, f]_\star$ , with  $\Omega$  given by (4.22). If we make explicit the dependence on  $\Gamma_i$  in  $C$  as

$$C(x|y; \Gamma_i) = C_{\text{aux}}(x|y; \Gamma_1) + C_{\text{dyn}}(x|y; \Gamma_1) \Gamma_2, \quad (4.25)$$

we can see that  $D_0 C = 0$  splits into

$$D_L C_{\text{aux}} + \Gamma_1 [e, C_{\text{aux}}]_\star = 0, \quad D_L C_{\text{dyn}} + \Gamma_1 \{e, C_{\text{dyn}}\}_\star = 0, \quad (4.26)$$

where the Lorentz covariant derivative is  $D_L := d + [\omega, \bullet]_\star$ . It is known [73] that the equation for  $C_{\text{aux}}$  describes non-propagating degrees of freedom, and  $C_{\text{aux}}$  is indeed referred to as auxiliary. On the other hand, by expanding the equation for  $C_{\text{dyn}}$  in power series in  $y^\alpha$ , one can see that  $C_{\text{dyn}}$  contains two real  $AdS_3$  massless scalar fields together with all their on-shell nontrivial derivatives. At this stage, the next steps would be to

- (1) use  $\frac{\partial S_1^\alpha}{\partial z^\alpha} = C \star \kappa$  to solve  $S_{1\alpha}$  in terms of  $C$ , up to some pure-gauge contribution;
- (2) substitute for  $S_1^\alpha$  into  $\frac{\partial w_1}{\partial z^\alpha} = -D_0 S_{1\alpha}$  to solve the  $z$ -dependence of  $w_1$  in terms of  $C$ . One gets  $w_1 = \omega_1(x|y; \Gamma_i) + f_1(x, z|y; \Gamma_i)$ , where  $f_1$  is a known function linear in  $C$ ;
- (3) use  $D_0 \omega_1 = -\left(D_0 f_1\right)|_{z=0}$  as higher spin equation for  $\omega_1$ .

It is known [74], however, that the linearized curvature  $D_0 \omega_1$  does not receive nontrivial sources linear in  $C$ , and indeed the higher spin fields do not propagate. For a more complete and thorough analysis of the 3D equations, including the issue of field redefinition, locality and elimination of the auxiliary zero-form  $C_{\text{aux}}$  up to second order in the weak fields, see Ref. [75]. See also Ref. [76] for another review and exact solutions of the PV equations.

**Massive vacua.** Let us study the model around a different vacuum, namely  $B_0 = \nu \Gamma$ , where  $\nu$  is a constant and  $\Gamma$  is some element generated by the  $\Gamma_i$ , obeying  $\Gamma^\dagger = \Gamma$  and  $\Gamma^2 = 1$ . The vacuum equations now read

$$\begin{aligned} dU_0 + U_0 \star U_0 &= 0, \quad [\Gamma, U_0] = 0, \quad [\tilde{z}_\alpha, \Gamma]_\star = 0 \\ [\tilde{z}_\alpha, U_0]_\star &= 0, \quad [\tilde{z}_\alpha, \tilde{z}_\beta]_\star = -2i \epsilon_{\alpha\beta} (1 - \nu \Gamma \kappa), \end{aligned} \quad (4.27)$$

where we defined  $\tilde{z}_\alpha := S_{0\alpha}$ , that obey<sup>12</sup>  $\{\tilde{z}_\alpha, \Gamma \kappa\}_\star = 0$  and the deformed oscillator algebra. The strategy is first to find a solution for  $\tilde{z}_\alpha$ , and then solve  $[\tilde{z}_\alpha, U_0]_\star = 0$  by finding other deformed variables  $\tilde{y}_\alpha$  that star

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<sup>12</sup>Since  $\tilde{z}_\alpha$  commutes with  $\Gamma$  and anticommutes with  $\kappa$  from the necessary bosonic projection.



commute with  $\tilde{z}_\alpha$ . To this end, let us introduce, as done in Ref. [8],

$$\sigma_\alpha := \nu \int_0^1 dt t e^{ityz} (y_\alpha + z_\alpha), \quad \tau_\alpha := \nu \int_0^1 dt (t-1) e^{ityz} (y_\alpha + z_\alpha), \quad (4.28)$$

with  $yz := y^\alpha z_\alpha$ . One can check that they obey the following relations:

$$\begin{aligned} [z_{[\alpha}, \sigma_{\beta]}]_\star &= -i\nu \epsilon_{\alpha\beta} \kappa, \quad [\sigma_\alpha, \sigma_\beta]_\star = 0, \quad \{\sigma_\alpha, \tau_\beta\}_\star = 0, \\ [z_\alpha, \tau_\beta]_\star &= \{\sigma_\alpha, y_\beta\}_\star, \quad \{y_{[\alpha}, \tau_{\beta]}]_\star = i\nu \epsilon_{\alpha\beta}, \quad [\tau_\alpha, \tau_\beta]_\star = 0. \end{aligned} \quad (4.29)$$

Let us make the following Ansatz for the deformed oscillators:

$$\tilde{z}_\alpha = X z_\alpha + Y \sigma_\alpha, \quad \tilde{y}_\alpha = A y_\alpha + B \tau_\alpha, \quad (4.30)$$

where  $X, Y, A$  and  $B$  are built out of gamma matrices  $\Gamma_i$ . By demanding

$$[\tilde{z}_\alpha, \tilde{z}_\beta]_\star = -2i\epsilon_{\alpha\beta}(1 - \nu\Gamma\kappa), \quad [\tilde{y}_\alpha, \tilde{z}_\beta]_\star = 0, \quad (4.31)$$

one has the following constraints,

$$X^2 = 1, \quad XY = YX = -\Gamma, \quad [A, X] = \{A, Y\} = [B, X] = \{B, Y\} = 0. \quad (4.32)$$

As in the original Prokushkin–Vasiliev model, we demand that  $\tilde{y}_\alpha$  obey a deformed oscillator algebra:

$$[\tilde{y}_\alpha, \tilde{y}_\beta]_\star = 2i\epsilon_{\alpha\beta}(1 - \nu\Gamma), \quad \{\tilde{y}_\alpha, \Gamma\} = 0. \quad (4.33)$$

This imposes

$$\{A, \Gamma\} = 0, \quad A^2 = 1, \quad AB = -BA = -\Gamma. \quad (4.34)$$

A convenient solution, that admits propagating degrees of freedom, is given by  $\Gamma = \Gamma_{1234}$

$$X = 1, \quad Y = -\Gamma, \quad A = \Gamma_1, \quad B = -\Gamma_{234} = -\Gamma_1\Gamma,$$

where we have chosen the number of generators of the Clifford algebra to be  $N = 4$  and  $\Gamma_{i_1 \dots i_k} := \Gamma_{[i_1} \dots \Gamma_{i_k]}$ .

With this solution, fields obeying  $[f, \tilde{z}_\alpha]_\star = 0$  and  $[f, \Gamma] = 0$  are given by  $f(x|\tilde{y}; \Gamma_{ij})$ , i.e. star functions of the deformed  $\tilde{y}$ 's and of all bilinears  $\Gamma_{ij}$ , together with  $\Gamma$  itself, that is generated by bilinears. One can check that undeformed  $sp(2, \mathbb{R})$  is still generated by  $\tilde{y}$ 's as  $T_{\alpha\beta} := \frac{1}{4i} \{\tilde{y}_\alpha, \tilde{y}_\beta\}_\star$ , and one can take the background connection  $U_0 = \Omega$  to be

$$\Omega(x|\tilde{y}; \Gamma_{ij}) = \frac{1}{4i} \left( \omega^{\alpha\beta}(x) \tilde{y}_\alpha \star \tilde{y}_\beta + i\Gamma_{23} e^{\alpha\beta}(x) \tilde{y}_\alpha \star \tilde{y}_\beta \right) = \omega + i\Gamma_{23} e, \quad (4.35)$$

such that  $d\Omega + \Omega^{\star 2} = 0$  is solved by the  $AdS_3$  background. Fluctuations  $B_1$  defined by  $B = \nu\Gamma + B_1 + \dots$  obey the linearized equations

$$[\tilde{z}_\alpha, B_1]_\star = 0 \quad \Rightarrow \quad B_1 = C(x|\tilde{y}; \Gamma_{ij}), \quad D_0 C = 0, \quad (4.36)$$

where, as before,  $D_0 f := df + [\Omega, f]_\star$ , but now with  $\Omega$  given by (4.35). In order to find the propagating degrees of freedom, let us explicitate the  $\Gamma_{ij}$  dependence of  $C$ :

$$C(x|\tilde{y}; \Gamma_{ij}) = C_{\text{aux}}(x|\tilde{y}; \Gamma, i\Gamma_{23}) + C_{\text{dyn}}(x|\tilde{y}; \Gamma, i\Gamma_{23})i\Gamma_{24} , \quad (4.37)$$

and use it in the equation  $D_0 C = 0$ , obtaining

$$D_L C_{\text{aux}} + i\Gamma_{23} [e, C_{\text{aux}}]_\star = 0 , \quad D_L C_{\text{dyn}} + i\Gamma_{23} \{e, C_{\text{dyn}}\}_\star = 0 , \quad (4.38)$$

from which one can find that  $C_{\text{dyn}}$  contains four real massive propagating scalars, while  $C_{\text{aux}}$  yields four topological deformations. The analysis is now equivalent to the original Prokushkin–Vasiliev model, since at this level one can identify

$$(k)_{\text{PV}} = \Gamma , \quad (\nu)_{\text{PV}} = -\nu , \quad (\rho)_{\text{PV}} = \Gamma_1 , \quad (y_\alpha)_{\text{PV}} = \Gamma_1 y_\alpha , \quad (z_\alpha)_{\text{PV}} = \Gamma_1 z_\alpha , \quad (4.39)$$

$$(\psi_1)_{\text{PV}} = i\Gamma_{23} , \quad (\psi_2)_{\text{PV}} = i\Gamma_{24} . \quad (4.40)$$

One can truncate the master fields as

$$\Pi_\Gamma^+ A = A , \quad \Pi_\Gamma^+ B = B , \quad (4.41)$$

with the projector  $\Pi_\Gamma^+ := \frac{1+\Gamma}{2}$ , yielding a model with two propagating scalars contained in

$$C_{\text{dyn}}(x|\tilde{y}; \Gamma, i\Gamma_{23}) = \frac{1+\Gamma}{2} C_+(x|\tilde{y}; i\Gamma_{23}) . \quad (4.42)$$

**Minimal truncation.** We now want to further truncate the matter sector contained in  $C_+$  to a single, real scalar field. By using the anti-automorphism defined by

$$\tau(f \star g) = (-1)^{\deg(f)\deg(g)} \tau(g) \star \tau(f) , \quad (4.43)$$

$$\tau(z^\alpha, dz^\alpha; y^\alpha, \Gamma_i) = (-iz^\alpha, -idz^\alpha; iy^\alpha, \epsilon_{(i)}\Gamma_i) , \quad \epsilon_{(i)} = (+, +, -, -) , \quad (4.44)$$

one can truncate further, by requiring

$$\tau(A) = -A , \quad \tau(B) = B . \quad (4.45)$$

This last truncation leaves a single real propagating scalar in the spectrum, since now the two chains of fields multiplying 1 and  $i\Gamma_{23}$  only contain even and odd numbers of derivatives of the physical field:

$$C_+(x|\tilde{y}; i\Gamma_{23}) = \sum_{n=0}^{\infty} C_{\alpha_1 \dots \alpha_{4n}}^0(x) \tilde{y}^{\alpha_1} \dots \tilde{y}^{\alpha_{4n}} + i\Gamma_{23} \sum_{n=0}^{\infty} C_{\alpha_1 \dots \alpha_{4n+2}}^1(x) \tilde{y}^{\alpha_1} \dots \tilde{y}^{\alpha_{4n+2}} . \quad (4.46)$$

Finally, it is possible to find solution for the deformed oscillator algebra such that the deformed oscillators obey the same reality conditions and transformation properties under the tau map as the undeformed  $y^\alpha$  and  $z^\alpha$ , and therefore it is understood above that the master fields are expanded in terms of these deformed oscillators.

## 4.2 Covariant Hamiltonian action

In this section we begin by discussing some generalities on covariant Hamiltonian actions on  $\mathcal{X}_4 \times \mathcal{Z}_2$ . We then determine the constraints on the Hamiltonian such that it leads to a master action in which the master field content, including the Lagrange multipliers, are extended to consist of sum of even and odd forms of appropriate degree, and central elements. This action yields a generalized version of the PV field equations.

**Generalities.** In order to formulate the theory within the AKSZ framework [34] using its adaptation to noncommutative higher spin geometries proposed in [35], we assume a formulation of the PV system that treats  $\mathcal{Z}_2$  as being closed and introduce an open six-manifold  $\mathcal{M}_6$  with boundary

$$\partial\mathcal{M}_6 = \mathcal{X}_3 \times \mathcal{Z}_2 , \quad (4.47)$$

where  $\mathcal{X}_3$  is a closed manifold containing  $\mathcal{M}_3$  as an open submanifold. On  $\mathcal{M}_6$ , we introduce a two-fold duality extended [77, 6, 49]<sup>13</sup> set of differential forms given by

$$A = A_{[1]} + A_{[3]} + A_{[5]} , \quad B = B_{[0]} + B_{[2]} + B_{[4]} , \quad (4.48)$$

$$T = T_{[4]} + T_{[2]} + T_{[0]} , \quad S = S_{[5]} + S_{[3]} + S_{[1]} , \quad (4.49)$$

valued in  $\mathcal{A} \otimes \mathcal{C}_4^+$ ,  $\mathcal{C}_4^+$  being the even subspace of the four dimensional Clifford algebra, and where the subscript denotes the form degree. The restriction to the even Clifford subalgebra, i.e. to fields obeying  $[f(\Gamma_i), \Gamma] = 0$ , is required by demanding integrability of the field equations coming from the action that we will present in the following. We let  $\{J^I\}$  denote the generators of the ring of off-shell closed and central terms, i.e. elements in the de Rham cohomology of  $\mathcal{M}_6$  valued in the center of  $\mathcal{A} \otimes \mathcal{C}_4^+$ , which hence obey

$$dJ^I = 0 , \quad [J^I, f]_\star = 0 , \quad (4.50)$$

(off-shell) for any differential form  $f$  on  $\mathcal{M}_6$  valued in  $\mathcal{A} \otimes \mathcal{C}_4^+$ . Following the approach of [6], we consider actions of the form

$$S_H = \int_{\mathcal{M}_6} \text{Tr}_{\mathcal{A} \otimes \mathcal{C}_4^+} [S \star DB + T \star F + \mathcal{V}(S, T; B; J^I)] \quad (4.51)$$

$$= \int_{\mathcal{M}_6} \text{Tr}_{\mathcal{A} \otimes \mathcal{C}_4^+} [S \star dB + T \star dA - \mathcal{H}(S, T; A, B; J^I)] \quad (4.52)$$

where  $\text{Tr}_{\mathcal{A} \otimes \mathcal{C}_4^+}$  denotes a cyclic trace operation on  $\mathcal{A} \otimes \mathcal{C}_4^+$ . We assume a structure group gauged by  $A$  and that  $S$ ,  $T$  and  $B$  belong to sections, and (4.52) makes explicit the covariant Hamiltonian form, with

$$\mathcal{H}(S, T; A, B; J^I) = -S \star [A, B]_\star - T \star A \star A - \mathcal{V}(S, T; B; J^I) . \quad (4.53)$$

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<sup>13</sup>Starting from a universally Cartan integrable system and replacing each  $p$ -form by a sum of forms of degrees  $p, p+2, \dots, p+2N$ , and each structure constant by a function of off-shell closed and central terms, i.e. elements in the de Rham cohomology valued in the center of the fiber algebra, with a decomposition into degrees  $0, 2, \dots, 2N$ , yields a new universally Cartan integrable system, referred to as the  $N$ -fold duality extension of the original system. More generally, one may consider on-shell duality extensions by including on-shell closed complex-valued functionals into the extension of the structure constants [78, 49].

Thus, the coordinate and momentum master fields, defined by

$$(X^\alpha; P_\alpha) := (A, B; T, S) , \quad (4.54)$$

lie in subspaces of  $\mathcal{A}$  that are dually paired using  $\text{Tr}_{\mathcal{A}}$ , which leads to distinct models depending on whether these subspaces are isomorphic or not. In the reductions that follow, we shall consider the first type of models, moreover, for definiteness, we shall assume that

$$\mathcal{M}_6 = \mathcal{X}_4 \times \mathcal{Z}_2 , \quad (4.55)$$

and the associative bundle is chosen such that

$$\check{\mathcal{L}} = \oint_{\mathcal{Z}_2} \text{Tr}_{\mathcal{A} \otimes \mathcal{C}_4^+} [S \star DB + T \star F + \mathcal{V}(S, T; B; J^I)] , \quad (4.56)$$

is finite and globally defined on  $\mathcal{X}_4$ . The action can then be written as

$$S_{\text{H}} = \int_{\mathcal{X}_4} \check{\mathcal{L}} . \quad (4.57)$$

**The master action.** The Hamiltonian is constrained by gauge invariance, or equivalently, by universal on-shell Cartan integrability<sup>14</sup>. In addition, it is constrained by the requirement that the equations of motion on  $\mathcal{M}_6$  reduce to a desired set of equations of motion on  $\partial\mathcal{M}_6$  upon assuming natural boundary conditions. In order to obtain a model that admits consistent truncations to three-dimensional CS higher spin gravities, we need to assume that  $\mathcal{V}$  contains a term that is quadratic in  $T$ . The simplest possible such action is given by

$$S_{\text{H}} = \int_{\mathcal{M}_6} \text{Tr}_{\mathcal{A} \otimes \mathcal{C}_4^+} [S \star DB + T \star [F + g + h \star (B - \frac{1}{2}\mu \star T)] + \mu \star B \star S \star S] \quad (4.58)$$

where

$$g = g(J^I) , \quad h = h(J^I) , \quad \mu = \mu(J^I) \quad (4.59)$$

are even closed and central elements on  $\mathcal{M}_6$  in degrees

$$\deg(g, h, \mu) = (2 \bmod 2, 2 \bmod 2, 0 \bmod 2) . \quad (4.60)$$

The reality conditions are given by

$$(A, B; T, S; g, h, \mu)^\dagger = (-A, B; -T, S; -g, -h, -\mu) , \quad (4.61)$$

The total variation yields

$$\begin{aligned} \delta S_{\text{H}} &= \int_{\mathcal{M}_6} \text{Tr}_{\mathcal{A} \otimes \mathcal{C}_4^+} \left( \delta T \star \mathcal{R}^A + \delta S \star \mathcal{R}^B + \delta A \star \mathcal{R}^T + \delta B \star \mathcal{R}^S \right) \\ &+ \oint_{\partial\mathcal{M}_6} \text{Tr}_{\mathcal{A} \otimes \mathcal{C}_4^+} (T \star \delta A - S \star \delta B) , \end{aligned} \quad (4.62)$$

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<sup>14</sup> Covariant Hamiltonian actions are gauge invariant iff their equations of motion form universally Cartan integrable systems.

where the Cartan curvatures read

$$\begin{aligned}
\mathcal{R}^A &= F + g + h \star (B - \mu \star T) \approx 0 \\
\mathcal{R}^B &= DB + \mu \star [S, B]_\star \approx 0 \\
\mathcal{R}^T &= DT + [S, B]_\star \approx 0 \\
\mathcal{R}^S &= DS + h \star T + \mu \star S \star S \approx 0
\end{aligned} \tag{4.63}$$

The generalized Bianchi identities are

$$D\mathcal{R}^A \equiv h \star (\mathcal{R}^B - \mu \star \mathcal{R}^T), \tag{4.64}$$

$$D\mathcal{R}^B \equiv [(\mathcal{R}^A + \mu \star \mathcal{R}^S), B]_\star - \mu \star \{\mathcal{R}^B, S\}_\star, \tag{4.65}$$

$$D\mathcal{R}^T \equiv [\mathcal{R}^A, T]_\star + [\mathcal{R}^S, B]_\star - \{\mathcal{R}^B, S\}_\star, \tag{4.66}$$

$$D\mathcal{R}^S \equiv [\mathcal{R}^A, S]_\star + \mu \star [\mathcal{R}^S, S]_\star + h \star \mathcal{R}^T. \tag{4.67}$$

The gauge transformations

$$\delta_{\epsilon, \eta} A = D\epsilon^A - h \star (\epsilon^B - \mu \star \eta^T), \tag{4.68}$$

$$\delta_{\epsilon, \eta} B = D\epsilon^B - [\epsilon^A, B]_\star - \mu \star [\eta^S, B]_\star + \mu \star \{S, \epsilon^B\}_\star, \tag{4.69}$$

$$\delta_{\epsilon, \eta} T = D\eta^T - [\epsilon^A, T]_\star - [\eta^S, B]_\star + \{S, \epsilon^B\}_\star, \tag{4.70}$$

$$\delta_{\epsilon, \eta} S = D\eta^S - [\epsilon^A, S]_\star - \mu \star [\eta^S, S]_\star - h \star \eta^T, \tag{4.71}$$

which transform the Cartan curvatures into each other, induce

$$\delta_{\epsilon, \eta} S_H = \oint_{\partial \mathcal{M}_6} \text{Tr}_{A \otimes C_4^+} \left( \eta^T \star [F + g + h \star B] + \eta^S \star DB \right). \tag{4.72}$$

We take  $(\epsilon^B; \eta^T, \eta^S)$  to belong to sections of the structure group and impose<sup>15</sup>

$$(\eta^T, \eta^S)|_{\partial \mathcal{M}_6} = 0. \tag{4.73}$$

We have also assumed that  $(A, B)$  fluctuate on  $\partial \mathcal{M}_6$ , which implies

$$T|_{\partial \mathcal{M}_6} \approx 0 \approx S|_{\partial \mathcal{M}_6}. \tag{4.74}$$

The resulting boundary equations of motion

$$F + g + h \star B \approx 0, \quad DB \approx 0 \tag{4.75}$$

thus provide a duality extended version of the Prokushkin–Vasiliev equations, that is free from any interaction ambiguity, following a variational principle. Let us notice that in the action (4.58), the relative coefficient of the  $BSS$  and  $TT$  terms is fixed uniquely by Cartan integrability.

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<sup>15</sup> Following the AKSZ approach, the Batalin–Vilkovisky classical master equation requires that the ghosts corresponding to  $(\eta^T, \eta^S)$  vanish at  $\partial \mathcal{M}_6$  off-shell.

### 4.3 Consistent truncations

In this section we review [4] the consistent truncation of the above covariant Hamiltonian master action in six dimensions down to a  $BF$ -like model on  $\mathcal{X}_4$  that reproduces Blencowe's action. The truncation consists in integrating out the fluctuations in  $B$  around its vacuum expectation value followed by reductions on  $\mathcal{X}_4$ . Starting from the equations of motion (4.63) and setting  $B = 0$  yields

$$F + g - h \star \mu \star T = 0 , \quad DT = 0 , \quad (4.76)$$

and

$$DS + h \star T + \mu \star S \star S = 0 , \quad (4.77)$$

which together form a Cartan integrable system containing (4.76) as a subsystem, i.e. the free differential algebra generated by  $(A, T, S)$  contains a subalgebra generated by  $(A, T)$ . Assuming  $\partial\mathcal{M}_6$  to consist of a single component, it follows from  $S|_{\partial\mathcal{M}_6} = 0$  that  $S$  can be reconstructed from  $(A, T)$  on-shell<sup>16</sup> from (4.77). Therefore, the system (4.76) is a consistent truncation of the original system (4.63) on-shell.

Rewriting the full action (4.58) by integrating by parts in its  $SDB$ -term yields

$$S_H = \int_{\mathcal{M}_6} \text{Tr}_{\mathcal{A} \otimes \mathcal{C}_4^+} [T \star (F + g - \frac{1}{2} h \star \mu \star T) + B \star (DS + h \star T + \mu \star S \star S)] . \quad (4.78)$$

It follows that  $B = 0$  is a saddle point of the path integral at which  $B$  and  $S$  can be integrated out in a perturbative expansion. Schematically, modulo gauge fixing, one has

$$\int_{\langle B \rangle = 0} [DB][DS] e^{\frac{i}{\hbar} S_H} \sim e^{\frac{i}{\hbar} S_{\text{eff}}[A, T]} , \quad (4.79)$$

where the effective action

$$S_{\text{eff}}[A, T] = S_{\text{red}}[A, T] + O(\hbar) , \quad (4.80)$$

consists of loop corrections (comprising attendant functional determinants on noncommutative manifolds) and

$$S_{\text{red}} = \int_{\mathcal{M}_6} \text{Tr}_{\mathcal{A} \otimes \mathcal{C}_4^+} T \star (F + g - \frac{1}{2} h \star \mu \star T) . \quad (4.81)$$

The latter is a consistently reduced classical action in the sense that it reproduces the subsystem (4.76). The reduced system, which thus consists of the gauge sector of the original system, is a topological theory with local symmetries

$$\delta A = D\epsilon + \mu \star h \star \eta , \quad \delta T = D\eta - [\epsilon, T]_{\star} , \quad (4.82)$$

and equations of motion and boundary conditions given by

$$F + g - \mu \star h \star T = 0 , \quad DT = 0 , \quad (4.83)$$

$$T|_{\partial\mathcal{M}_6} = 0 . \quad (4.84)$$

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<sup>16</sup> Since  $T|_{\partial\mathcal{M}_6} = 0$  on-shell as well it follows that both  $S$  and  $T$  can be taken to vanish on  $\mathcal{M}_6$  on-shell.

The boundary equations are thus given by

$$(F + g)|_{\partial\mathcal{M}_6} = 0 . \quad (4.85)$$

To address Blencowe's theory, we truncate once more by reducing (4.76) under the assumptions that

$$g = 0 , \quad \mu = \mu_{[0]} \equiv \mu_0 , \quad h = J , \quad (4.86)$$

where  $\mu_0$  is an imaginary constant, and

$$A = \check{W}_{[1]} - \check{K}_{[1]} - \mu_0 J \star \check{K}_{[1]} , \quad T = \check{T}_{[2]} + \check{K}_{[1]} \star \check{K}_{[1]} - \mu_0 J \star \check{T}_{[2]} , \quad (4.87)$$

where we define checked fields to be  $z$ -independent:  $\check{f} \in \Omega(\mathcal{X}_4) \otimes \mathcal{A} \otimes \mathcal{C}_4^+$ . By defining

$$\check{F} = d_X \check{W} + \check{W} \star \check{W} , \quad \check{D}\check{K} = d_X \check{K} + [\check{W}, \check{K}]_\star , \quad \check{D}\check{T} = d_X \check{T} + [\check{W}, \check{T}]_\star , \quad (4.88)$$

and suppressing the subscripts indicating form degrees, the reduction of (4.76) yields

$$\check{F} + \check{T} = 0 , \quad \check{D}\check{T} = 0 , \quad (4.89)$$

$$D\check{K} - \check{K} \star \check{K} + \check{T} = 0 , \quad (4.90)$$

which is a Cartan integrable system containing (4.89) as a subsystem. From (4.84) and (4.87), we deduce the boundary conditions

$$\check{T}|_{\partial\mathcal{X}_4} = 0 = (\check{K} \star \check{K})|_{\partial\mathcal{X}_4} , \quad (4.91)$$

Substituting (4.87) into (4.81) and using (4.91) we obtain

$$\check{S}_{\text{red}}[\check{W}, \check{T}] = -\mu_0 \int_{\mathcal{X}_4} \int_{\mathcal{Z}_2} \text{Tr}_{\mathcal{A} \otimes \mathcal{C}_4^+} J \star \check{T} (\check{F} + \frac{1}{2}\check{T}) . \quad (4.92)$$

At this stage, we truncate the models further as follows:

$$\check{W} = \Pi_\Gamma^+ W_+ + \Pi_\Gamma^- W_- , \quad \check{T} = \Pi_\Gamma^+ T_+ + \Pi_\Gamma^- T_- , \quad (4.93)$$

where  $W_\pm$  and  $T_\pm$  are independent of  $\Gamma_i$ . Inserting (4.93) into (4.92) and using

$$\int_{\mathcal{Z}_2} \text{Tr}_{\mathcal{A} \otimes \mathcal{C}_4^+} J \star \Pi_\Gamma^\pm \check{f} =: \pm \frac{i\pi}{2} \text{STr}_{\text{Aq}(2)} \check{f} , \quad (4.94)$$

for  $\check{f}$  independent of  $\Gamma_i$  and belonging to  $\text{Aq}(2)$ , the associative algebra given by the universal enveloping algebra of the undeformed oscillators  $y^\alpha$ . We thereby obtain the following four-dimensional Hamiltonian extension of Blencowe's action:

$$S_{\text{Bl}} = -\frac{i\pi}{2} \mu_0 \int_{\mathcal{X}_4} \text{STr}_{\text{Aq}(2)} [T_+(F_+ + \frac{1}{2}T_+) - T_-(F_- + \frac{1}{2}T_-)] . \quad (4.95)$$

Assuming that  $\mathcal{X}_4 = \mathcal{X}_3 \times [0, \infty[$  and that all fields fall off at  $\mathcal{X}_3 \times \infty$ , and assuming furthermore that  $\mathcal{X}_3$  has a simple topology, the elimination of the Lagrange multipliers yields

$$S_{\text{BI}} = \frac{i\pi}{2} \mu_0 (S_{\text{CS}}[W_+] - S_{\text{CS}}[W_-]) , \quad (4.96)$$

with

$$S_{\text{CS}}[W] = \oint_{\mathcal{X}_3} \text{STr}_{\text{Aq}(2)} \left[ \frac{1}{2} W \star dW + \frac{1}{3} W \star W \star W \right] , \quad (4.97)$$

where now  $d$  denotes the exterior derivative on  $\mathcal{X}_3$ . Equivalently,

$$S_{\text{BI}} = i\mu_0\pi \oint_{\mathcal{X}_3} \text{STr}_{\text{Aq}(2)} \left[ E \star (d\Omega + \Omega \star \Omega) + \frac{1}{3} E \star E \star E \right] , \quad W_{\pm} = \Omega \pm E , \quad (4.98)$$

from which we identify

$$\mu_0 = -\frac{4i}{\pi^2} \frac{\ell_{\text{AdS}}}{G_{\text{N}}} \quad (4.99)$$

using the conventions of [2].

**Truncation to the Prokushkin–Segal–Vasiliev action.** In the paper [4] that we are reviewing here, another consistent truncation of the action (4.58) is shown to reproduce the action principle [79] for the PV equation that is formulated in a 2-dimensional base space for the  $z^\alpha$  oscillators. The interested reader can find all the details in [4].

## 5 Higher spins and topological strings

In this section we shall argue, based on the formal structure of higher spin equations and actions, that higher spin dynamics can be described in terms of first-quantized topological open strings.

Vasiliev’s equations exhibit two basic properties akin to open string field theory: First of all, we recall that in Cartan’s formulation of field theories, as free differential algebras, the spaces of forms in degrees one and zero correspond to gauge Lie algebras and spectra of local degrees of freedom, respectively. In this respect, a remarkable feature of Vasiliev’s higher spin gravities is that their one- and zero-form modules are unified into associative algebras<sup>17</sup>. Second, they can be embedded into a Frobenius–Chern–Simons theory with a cubic action principle that only refers to the trace and star product operations of the associative algebra and a nilpotent differential containing the de Rham differential. These algebraic structures can be naturally encoded into a class of two-dimensional Poisson sigma models with gauged supersymmetry, corresponding to the de Rham operator, which are our candidate topological open string models.

In order to explain the rational behind these models in more detail, let us start from a particle on a symplectic manifold  $N$  with symplectic potential one-form  $\vartheta = d\phi^\alpha \vartheta_\alpha$  and Hamiltonian  $H$ , consisting of

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<sup>17</sup> An interesting consequence is that the spectra of massless particles and generalized Type D solutions in four-dimensional higher spin gravity are related by a  $\mathbb{Z}_2$ -symmetry [48], unlike in ordinary gauge theories, in which the Type D solution spaces are finite dimensional.



constraints and Lagrange multipliers, as described by an action of the form

$$S_{\vartheta} = \int_C (\vartheta - H) ,$$

integrated along a worldline  $C$ . Assigning physical states of the system to Hilbert spaces it is natural to consider open worldlines whose boundaries are allowed to fluctuate in Lagrangian submanifolds of  $N$ . Alternatively, and more generally, one may assign the physical states to density matrices, in which case it is natural to take  $C$  to be closed, and choose boundary conditions such that the path integral provides a trace operation. Considering initially the case of a trivial Hamiltonian, and letting  $\Pi^{\alpha\beta}$  denote the Poisson bi-vector on  $N$ , *i.e.* the inverse of the symplectic two-form  $\omega_{\alpha\beta} = \partial_{\alpha}\vartheta_{\beta} - \partial_{\beta}\vartheta_{\alpha}$ , the resulting path integral over worldlines  $C$  can be extended into a path integral over open worldsheets  $\Sigma$  with boundary  $C$ , weighted by  $\exp \frac{i}{\hbar} S_{\Pi}$  where  $S_{\Pi}$  is the action of the Ikeda–Schaller–Strobl Poisson sigma model

$$S_{\Pi} = \int_{\Sigma} (\eta_{\alpha} \wedge d\phi^{\alpha} + \frac{1}{2} \Pi^{\alpha\beta} \eta_{\alpha} \wedge \eta_{\beta}) ,$$

subject to the boundary condition  $\eta_{\alpha}|_C = 0$ . (which can be derived on-shell using the variational principle but that actually must be imposed off-shell as well in order for the classical Batalin–Vilkovisky master equation to hold). The resulting path integral can be performed in two steps: First over discs with a point of  $C$  attached to a fixed point  $p_0 \in N$  (which one may think of as a  $(D-1)$ -brane), and then by integrating over  $p_0$  using the symplectic measure. Inserting vertex operators along points  $p_i \in C$ , given by (the pull-back of) functions  $f_i$  on  $N$ , the resulting path integral can be viewed as a formal definition of  $\text{Tr}_N \prod_i^{\star} f_i$ , where  $f \star g$  is an associative noncommutative product, referred to as the star product. Provided that  $f$  and  $g$  belong to sufficiently smooth classes of functions (*e.g.* polynomials or formal power series with coefficients given by power series in  $\hbar$ ), then the star product has an expansion in terms of pointwise derivatives, *viz.*

$$f \star g = fg + i\hbar\{f, g\} + \dots , \quad \{f, g\} := \Pi^{\alpha\beta} \partial_{\alpha} f \partial_{\beta} g ,$$

where the higher order terms in the  $\hbar$  expansion are given in terms of multiple derivatives of the functions and the Poisson structure (but not its inverse), while in order to compose more general elements, including (nonperturbative) distributions on  $N$  and (which arise as density matrices related to unitary representations), nonlocal versions of the star product, based on auxiliary integrations, are required. Unlike the particle action, the Poisson sigma model remains well-defined on Poisson manifolds, where  $\Pi^{\alpha\beta}$  is assumed to be a bi-vector obeying

$$\Pi^{\delta[\alpha} \partial_{\delta} \Pi^{\beta\gamma]} = 0 ,$$

as this ensures the invariance of the action under  $\eta_{\alpha}$  gauge transformations, while there is no requirement on its invertibility.

Historically, the existence of a star product on general symplectic manifolds was first established formally by De Wilde and Lecomte [80]. Its pointwise form was given explicitly and on a manifestly covariant form by Fedosov [81]. His construction resembles that of Vasiliev, though it does not provide any dynamics, which instead will require a suitable gauging leading to topological open strings, as we shall propose below. Later,

Kontsevich used his formality theorem to show the existence and uniqueness (up to similarity transformations and changes of basis corresponding to different ordering schemes) of the star product on general Poisson manifolds. He also provided an explicit formula in the case of  $N \cong \mathbb{R}^n$ , derived soon after by means of perturbative AKSZ quantization of the Poisson sigma model by Cattaneo and Felder [82], who also provided a globally defined star product for any  $(N, \Pi)$  in [83].

In order to spell out our proposal, we need one more basic ingredient of Vasiliev's models and their FCS generalizations. In addition to the aforementioned fusion of their Cartan modules into associative fiber algebras, their simplicity relies on yet one more unification, namely of spacetime and noncommutative symplectic manifolds into extended base manifolds of Poisson type so as to facilitate the construction of closed and central two-forms whose star products with the zero-form provide (nontrivial) co-cycles for the curvatures. The resulting framework is that of differential Poisson manifolds, which are natural generalizations of Poisson manifolds on which classes of differential forms (and distributions) in different degrees, and not just functions, can be equipped by star products by deforming the classical wedge product along differential Poisson brackets. In the case of sufficiently smooth objects, the resulting pointwise star product reads

$$f \star g = f \wedge g + i\hbar\{f, g\} + \cdots, \quad (5.1)$$

where the differential Poisson bracket, that now involves the bi-vector as well as a compatible connection, is of the form<sup>18</sup>

$$\{\omega, \eta\} = \Pi^{\alpha\beta} \nabla_\alpha \omega \wedge \nabla_\beta \eta + (-1)^{\deg(\omega)} \tilde{R}^{\alpha\beta} \wedge i_\alpha \omega \wedge i_\beta \eta, \quad (5.2)$$

where  $\tilde{R}^{\alpha\beta} = \Pi^{\alpha\gamma} \tilde{R}^\alpha_\gamma$  and  $\tilde{R}^\alpha_\beta$  is the curvature of the connection shifted by the torsion. When written on the above form, the bracket is graded skew-symmetric, of vanishing intrinsic degree, compatible with the de Rham differential and obeying Leibniz rule, while the graded Jacobi identity requires the additional conditions<sup>19</sup>

$$\Pi^{\delta[\alpha} T_{\delta\epsilon}^{\beta]} \Pi^{\gamma]\epsilon} = 0, \quad \Pi^{\alpha\rho} \Pi^{\sigma\beta} R_{\rho\sigma}{}^\gamma{}_\delta = 0, \quad (5.3)$$

$$\Pi^{\alpha\lambda} \nabla_\lambda \tilde{R}_{\beta\gamma}{}^{\rho\sigma} = 0, \quad \tilde{R}_{\epsilon[\rho}{}^{(\alpha\beta} \tilde{R}_{\sigma\lambda]}{}^{\gamma)\epsilon} = 0. \quad (5.4)$$

By extending the formality theorem to the graded supermanifold with coordinates  $(\phi^\alpha, \theta^\alpha)$  (see [46] for a related discussion), the  $\hbar$ -expansion of the star product can be determined together with a nilpotent operator, given by an  $\hbar$ -deformation of the de Rham differential, by requiring associativity and compatibility. In particular, in degree zero, the star product must yield Cattaneo and Felder's manifestly covariant form of Kontsevich product (which in its turn reproduces Fedosov's product in the symplectic case).

Motivated by the above considerations, it was proposed in [1] that the differential star product algebra can be obtained by perturbative quantization of the supersymmetric two-dimensional topological sigma model based on the (classical) action

$$S = \int_\Sigma \left( \eta_\alpha \wedge d\phi^\alpha + \chi_\alpha \wedge \nabla \theta^\alpha + \frac{1}{2} \Pi^{\alpha\beta} \eta_\alpha \wedge \eta_\beta + \frac{1}{4} \theta^\alpha \theta^\beta \tilde{R}_{\alpha\beta}{}^{\gamma\delta} \chi_\gamma \wedge \chi_\delta \right). \quad (5.5)$$

<sup>18</sup>Strictly speaking, there remains one possible tensorial deformation; see [1] for a more detailed form of the Poisson bracket.

<sup>19</sup>The quadratic constraint on the curvature tensor is a Yang-Baxter equation and the two-dimensional differential Poisson sigma model can thus be used as a framework for associative algebras including Hopf algebras.

Indeed, the conditions on the background fields required by the Jacobi identity, as listed above, can equivalently be derived by requiring the appropriate two-dimensional gauge symmetries, which serve to gauge away all local degrees of freedom except the constant modes in  $\phi^\alpha$  and  $\theta^\alpha$ . Moreover, the coefficient of the four-fermi coupling is fixed by the requirement that the action has a global nilpotent supersymmetry given by

$$\delta_f \phi^\alpha = \theta^\alpha, \quad \delta_f \theta^\alpha = 0, \quad (5.6)$$

$$\delta_f \eta_\alpha = \frac{1}{2} \widetilde{R}_{\beta\gamma}{}^\delta{}_\alpha \chi_\delta \theta^\beta \theta^\gamma - \Gamma_{\alpha\beta}^\gamma \eta_\gamma \theta^\beta, \quad \delta_f \chi_\alpha = -\eta_\alpha + \Gamma_{\alpha\beta}^\gamma \chi_\gamma \theta^\beta. \quad (5.7)$$

This transformations can be identified as an avatar for the de Rham differential upon representing the forms on  $N$  as functions on the parity shifted bundle  $T[1]N$  coordinatized by  $(\phi^\alpha, \theta^\alpha)$ . It can be shown that the model can be reformulated on more general  $(n|n)$  supermanifolds equipped with super Poisson bi-vectors. The global symmetries of this model, which include the original Hamiltonian vector fields as well as more general super Killing vectors, such as  $\delta_f$ , can be gauged.

The special case of gauging of  $\delta_f$  was studied in [13]. It remains to be examined whether it is consistent at the quantum level, which may require extra conditions on differential Poisson geometry (*e.g.* its Ricci tensor), though the absence of obstructions at the first sub-leading order in  $\hbar$  [46] suggests their absence to all orders in perturbation theory. Assuming quantum consistency, it is natural to propose that finite deformations of the background can be modelled by a topological open string field  $\Psi$  obeying

$$\{Q, \Psi\}_\star + \Psi \star \Psi = 0, \quad (5.8)$$

where the BRST operator  $Q$  contains a sector that gauges the de Rham differential acting on the zero-modes of  $\phi^\alpha$ . We claim that the FCS formulation of four-dimensional higher spin gravity in twistor space can be obtained by a reduction of the above system down to a finite set of modes describing stretched strings, whereas in order to obtain the minimal bosonic models<sup>20</sup> based on vector oscillators one has to gauge the additional  $sp(2)$  Killing vectors generated by the moment functions of the conformal particle.

Finally, let us remark on the connection to tensionless strings in anti-de Sitter spacetime. To this end, it is important that the associative algebras in higher spin gravity are of a special type related to singletons. These algebras can be realized either via the group algebras (or enveloping algebras) of the underlying finite-dimensional isometry groups or oscillator algebras over ideals given by singleton annihilators<sup>21</sup>, which can be thought of as the equations of motions for conformal particles on the embedding space of anti-de Sitter spacetime. Thus one may view the topological open string as the germ of a tensionless string, consisting of two string partons orbiting (as conformal particles) around a center-of-mass. Indeed, similar excitations, carrying the quantum numbers of singletons, are known to arise in the form of cusps on tensionful closed

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<sup>20</sup>These models are known as the  $D$ -dimensional Type A models. The  $D$ -dimensional Type B models should arise in a similar fashion from gauging an  $osp(1|2)$  algebra.

<sup>21</sup>The topological open string approach may be useful in further elucidating the whether there exist two dual underlying first-quantized formulations, one on the group manifold, thought of as a Poisson manifold, and another one directly on the singleton phase space, thought of as a symplectic manifold.

boconic strings in  $D$ -dimensional anti-de Sitter space time in the semi-classical limit (described by soliton solutions of the of the Nambu–Goto action). Thus, as proposed in [41], one may think of the Hagedorn transition in flat spacetime as switching on a small negative cosmological constant whereby the cyclically ordered one-string states of tensionful strings break up in the tensionless limit into totally symmetric multi-singleton states. In particular, on physical grounds, long (folded) string states, which thus connect two cusps at the opposite side of the center-of-mass (with closed worldlines in periodic anti-de Sitter spacetime), should remain self-interacting in this limit, at least in the classical limit of the string field theory, and admit a first-quantized description as an  $sp(2)$ -gauged topological open bosonic string with an effective field theory description in terms of minimal bosonic higher spin gravity.

In summary, one further piece of motivation for studying topological open strings of the type proposed above is thus that they may provide a link between higher spin gravity and tensionless strings in anti-de Sitter spacetime, and possibly also a glimpse of what one might expect from a background independent formulation of closed string field theory.

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